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# EXAMPLES IN THE METHODS OF MODERN GEOMETRY; ESPECIALLY TRILINEAR COORDINATES.

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# A COLLECTION OF PROBLEMS AND THEOREMS,

WITH

HINTS, RESULTS, AND OCCASIONAL SOLUTIONS,

FORMING

EXAMPLES IN THE METHODS

OF

# MODERN GEOMETRY;

ESPECIALLY

# TRILINEAR COORDINATES.

BY THE

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# PREFACE.

About five years ago, the author of the following pages suggested, how advisable it would be, to draw up a treatise, confined entirely to the modern method of Trilinear Coordinates, and inasmuch as many complaints had been made, that the valuable Treatises already published on Analytical Geometry, contained only confused and incomplete chapters, made up of brief investigations of the beautiful methods adopted in that system; he presumed to take upon himself the task; but, upon the eve of the manuscript being completed, he was anticipated by the announcement of a work of the kind by the Rev. N. M. Ferrers.

The "desideratum" then being in such able hands, the author at once relinquished the publication of his compilation; but having since been solicited to draw up a collection of examples contained in the original manuscript, with hints, results, and occasional solutions; he has ventured to offer the result of his labours to those students who feel an interest in this subject, introducing such questions of a new character as may occur in some of the most useful existing periodicals, so remarkable for their valuable and original contributions.

The author can therefore only advance this apology to Mathematical students for laying before them this production, and trusts that while they may indulge him, by not scanning with too critical an eye, any unavoidable inaccuracies; they will, at least, give him some small share of credit for having made an attempt to attract the attention of the aspiring student to a valuable and interesting branch of the science, which has lately been the subject of deep research by many of the most able Mathematicians of the present day.

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## CORRIGENDA.

Page 2, line 15, *instead of* $\alpha'\beta'$ and $\alpha''\beta''$ *read* $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$.
" 22, " 14, " diagonal *read* quadrilateral.
" 31, " 23, " $wb^2 + va^2 - 2u'ab$ *read* $ub^2 + va^2 - 2w'ab$.

# MODERN GEOMETRY.

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## CHAPTER I.

### EXAMPLES ON THE STRAIGHT LINE REFERRED TO TRILINEAR COORDINATES.

1. THE Triangle of Reference.

In the Cartesian theory the position of a point is determined by means of its distances from two given straight lines called coordinate axes, but in the trilinear theory the point is determined by the ratios of its distances from three given straight lines in that plane which do not pass through the same point. These three lines form a triangle which is called the *Triangle of Reference.*

2. If $P$ be any point situated in a plane referred to two straight lines $AO$ and $OB$, where the perpendicular distance of $P$ from $AO$ is $\alpha$, and the perpendicular distance of $P$ from $BO$ is $\beta$, the equation to $OP$ will be represented by

$$\alpha \pm k\beta = 0,$$

according as $P$ is considered outside of the acute angle at $O$ or inside.

3. By means of the last, prove that the angular bisectors of any triangle pass through the same point.

4. If $\alpha$, $\beta$, $\gamma$ represent the trilinear coordinates of any point within the triangle of reference, of which $\Delta$ represents the area,

$$a\alpha + b\beta + c\gamma = 2\Delta.$$

Prove this in all its various cases.

5. If $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$ be the coordinates of two points, $r$ the distance between them,

$$r^2 = \frac{(\beta_1 - \beta_2)^2 + (\gamma_1 - \gamma_2)^2 + 2(\beta_1 - \beta_2)(\gamma_1 - \gamma_2)\cos A}{\sin^2 A}.$$

6. Shew also that

$$r^2 = \frac{abc}{4\Delta^2}\{a\cos A\,(\alpha_1 - \alpha_2)^2 + b\cos B\,(\beta_1 - \beta_2)^2 + c\cos C\,(\gamma_1 - \gamma_2)^2\}.$$

7. The general equation to a straight line may be put under the form

$$l\alpha + m\beta + n\gamma = 0.$$

8. The equation to a straight line passing through two given points $\alpha'\beta'$ and $\alpha''\beta''$ is ✓

$$\alpha(\beta'\gamma'' - \gamma'\beta'') + \beta(\gamma'\alpha'' - \gamma''\alpha') + \gamma(\alpha'\beta'' - \alpha''\beta') = 0.$$

9. If $l\alpha + m\beta + n\gamma = 0$ and $l'\alpha + m'\beta + n'\gamma = 0$ be the equations to two lines, then will the equation to the line passing ✓ through their point of intersection be

$$l\alpha + m\beta + n\gamma = k(l'\alpha + m'\beta + n'\gamma),$$

where $k$ is an arbitrary constant.

10. The condition that three points whose coordinates are respectively $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, $\alpha_3\beta_3\gamma_3$ may lie on the same straight line is ✓

$$\alpha_1\beta_2\gamma_3 - \alpha_1\beta_3\gamma_2 + \alpha_2\beta_3\gamma_1 - \alpha_2\beta_1\gamma_3 + \alpha_3\beta_1\gamma_2 - \alpha_3\beta_2\gamma_1 = 0.$$

11. If $\alpha = 0$, $\beta = 0$, $\gamma = 0$ represent the three sides of a triangle respectively, shew that the sides of any triangle which has one angle on each side of the former may be represented by

$$\alpha + n\beta + \frac{\gamma}{m} = 0, \quad \frac{\alpha}{n} + \beta + l\gamma = 0, \quad m\alpha + \frac{\beta}{l} + \gamma = 0,$$

where $l$, $m$, $n$ are arbitrary constants.

Shew also, that if the lines joining corresponding angles of the two triangles pass through the same point

$$lmn = -1.$$

12. The three equations to the lines joining the angular points with the middle points of the opposite sides are

$$\alpha \sin A - \beta \sin B = 0,$$

$$\beta \sin B - \gamma \sin C = 0,$$

$$\gamma \sin C - \alpha \sin A = 0.$$

Hence these three lines intersect in the same point.

13. The equations to the three perpendiculars let fall from the angular points of a triangle upon the opposite sides are

$$\alpha \cos A - \beta \cos B = 0,$$

$$\beta \cos B - \gamma \cos C = 0,$$

$$\gamma \cos C - \alpha \cos A = 0.$$

Hence these three lines intersect in the same point.

14. The condition that two straight lines, whose equations are

$$l\alpha + m\beta + n\gamma = 0,$$

and $$l'\alpha + m'\beta + n'\gamma = 0,$$

may be parallel to one another, is

$$(mn' - m'n) \sin A + (nl' - ln') \sin B + (lm' - l'm) \sin C = 0.$$

*Note.* If two lines be parallel to each other, their equations differ only by a constant.

Thus, if $\gamma = 0$ be the equation to the third side of a triangle, the equation to a line through $C$ and parallel to $AB$ will be

$$\alpha \sin A + \beta \sin B = 0.$$

For since the line will be represented by

$$\gamma + k(\alpha \sin A + \beta \sin B + \gamma \sin C) = 0,$$

when $$\left.\begin{matrix}\alpha = 0\\ \beta = 0\end{matrix}\right\}, \quad \gamma = -\alpha \sin B;$$

therefore $$\alpha \sin A + \beta \sin B = 0$$

is the equation required.

Again, if $E$ and $F$ be the middle points of the side $ABC$ of any triangle, the equation to $EF$ may be found from the equation

$$\alpha + k(\alpha \sin A + \beta \sin B - \gamma \sin C) = 0,$$

when $$\left.\begin{matrix}\beta = -\dfrac{C}{2}\sin A\\ \gamma = 0\\ \alpha = -\dfrac{C}{2}\sin B\end{matrix}\right\}, \quad k = -\frac{1}{2\sin A}.$$

Therefore, for $EF$, we have

$$-\alpha \sin A + \beta \sin B + \gamma \sin C = 0.$$

15. The condition that the straight line represented by

$$l\alpha + m\beta + n\gamma = 0$$

may be perpendicular to the line represented by

$$l'\alpha + m'\beta + n'\gamma = 0$$

is $ll' + mm' + nn'$

$$= (m'n + mn')\cos A + (l'n + n'l)\cos B + (l'm + m'l)\cos C.$$

16. The three perpendiculars from the middle points of the sides of a triangle will be represented by

$$\beta \sin B - \gamma \sin C + \alpha \sin(B - C) = 0,$$

$$\gamma \sin C - \alpha \sin A + \beta \sin(C - A) = 0,$$

$$\alpha \sin A - \beta \sin B + \gamma \sin(A - B) = 0.$$

*Note.* If we multiply these three equations respectively by

$$\sin 2A, \ \sin 2B, \text{ and } \sin 2C,$$

we find, by a little reduction, their sum will vanish.

Therefore, the three lines will intersect in a point.

17. The inclination between two straight lines, whose equations are

$$l\alpha + m\beta + n\gamma = 0,$$

and $$l'\alpha + m'\beta + n'\gamma = 0,$$

will be determined by

$$\theta - \theta' = \tan^{-1} \frac{(mn' - nm') \sin A + (l'n - n'l) \sin B + (lm' - l'm) \sin C}{ll' + mm' + nn' - (m'n + n'm) \cos A - (l'n + n'l) \cos B - (lm' + l'm) \cos C}.$$

Cor. 1. If $\theta - \theta' = \frac{\pi}{2}$, which is the inclination when the lines are perpendicular to each other, we have

$$ll' + mm' + nn'$$
$$- (mn' + m'n) \cos A - (l'n + n'l) \cos B + (lm' + l'm) \cos C = 0.$$

Cor. 2. If $\theta - \theta' = 0$ the lines will be parallel and

$$(mn' - nm') \sin A + (l'n - n'l) \sin B + (lm' - l'm) \sin C = 0.$$

18. The perpendicular distance of a given point $\alpha'\beta'\gamma'$ from the straight line whose equation is

$$l\alpha + m\beta + n\gamma$$

is $$\delta = \frac{l\alpha' + m\beta' + n\gamma'}{\{l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C\}^{\frac{1}{2}}}.$$

19. A straight line through $\alpha = 0$, $\beta = 0$, and perpendicular to the line represented by $\beta = 0$, will be determined by

$$\alpha + \beta \cos C = 0.$$

20. If a straight line $DE$ cuts the sides $AB$ and $AC$, its equation will be of the form

$$l\alpha + m\beta + n\gamma = 0,$$

and if another line perpendicular to this line be drawn through $A$, its equation will be

$$(l \cos B + m \cos A - n) \beta - (l \cos C - m + n \cos A) \gamma = 0.$$

21. If $AO$, $BO$, and $CO$ be the lines joining the angular points of a triangle and the center of the circumscribing circle, their equations will be

$$\begin{aligned} \text{for } AO \quad & \beta\cos C - \gamma\cos B = 0, \\ BO \quad & \gamma\cos A - \alpha\cos C = 0, \\ CO \quad & \alpha\cos B - \beta\cos A = 0. \end{aligned}$$

Cor. The given condition is that these three lines pass through the same point.

Multiply the first equation by $\cos A$, the second by $\cos B$, and the third by $\cos C$, we find, by addition, that their sum vanishes; hence they pass through the same point.

22. The equation to the lines passing through the centre of the inscribed circle perpendicular to the side of the triangle whose equation is $\alpha = 0$, $\beta = 0$, $\gamma = 0$ will be respectively

$$\begin{aligned} \alpha(\cos C - \cos B) + \beta(1 + \cos B) - \gamma(1 + \cos C) &= 0, \\ \alpha(1 + \cos A) + \beta(\cos C - \cos A) - \gamma(1 + \cos C) &= 0, \\ -\alpha(1 + \cos A) + \beta(1 + \cos B) + \gamma(\cos A - \cos B) &= 0. \end{aligned}$$

23. The equation to the line passing through the centers of the inscribed and circumscribed circles will be

$$\alpha(\cos B - \cos C) + \beta(\cos C - \cos A) + \gamma(\cos A - \cos B) = 0.$$

24. The equation to the line joining $A$ with the point in which the line bisecting the angle $C$ meets a line through the point $B$ and parallel to $CA$ is

$$\beta\sin A + \gamma\sin C = 0.$$

25. If $ABC$ be a triangle, $AO$, $BO$, $CO$ the lines joining the angular points and the center of the circumscribing circle, shew that the equations of the following lines will be

for $AG$ parallel to $BO$ is $\beta\cos C + \gamma\cos(A - C) = 0$,
$BH$ ............ $CO$ ... $\gamma\cos A + \alpha\cos(A - B) = 0$,
$CK$ ............ $AO$ ... $\alpha\cos B + \beta\cos(B - C) = 0$,
$AL$ ............ $CO$ ... $\beta\cos(A - B) + \gamma\cos B = 0$,
$BM$ ............ $AO$ ... $\gamma\cos(B - C) + \alpha\cos C = 0$,
$CN$ ............ $BO$ ... $\alpha\cos(A - C) + \beta\cos A = 0$.

26. The equations to the three sides of the triangle formed by joining the feet of the perpendiculars let fall from the angular points upon the opposite sides are respectively

$$\alpha \cos A + \beta \cos B - \gamma \cos C = 0,$$
$$\beta \cos B + \gamma \cos C - \alpha \cos A = 0,$$
$$\gamma \cos C + \alpha \cos A - \beta \cos B = 0.$$

27. The equations to the three sides of the triangle formed by joining the middle points of the three sides of a triangle are respectively

$$\alpha \sin A + \beta \sin B - \gamma \sin C = 0,$$
$$\beta \sin B + \gamma \sin C - \alpha \sin A = 0,$$
$$\gamma \sin C + \alpha \sin A - \beta \sin B = 0.$$

28. If lines be drawn through $C$, $A$, and $B$, viz. $CR$, $AP$, $BQ$ respectively parallel to $DE$, $EF$, and $DF$ their equations will be

$$\text{for } CR \quad \alpha \sin A + \beta \sin B = 0,$$
$$AP \quad \beta \sin B + \gamma \sin C = 0,$$
$$BQ \quad \gamma \sin C + \alpha \sin A = 0.$$

29. In question 26 apply the property of question 11, viz.

$$lmn = -1.$$

30. Let there be two inscribed triangles to a primary triangle, the one formed by joining the middle points of the sides of the primary triangle, and the other by joining the feet of the perpendiculars from the angular points of the primary triangle upon the opposite sides; then, if lines be drawn through the angular points of the primary triangle parallel to either of the inscribed triangles, these lines will meet the sides of the other inscribed triangle produced, in three points, which points will range in the same straight line. (*Vide* Remarks on this Chapter).

Making use of the same notation as in 26, 27, 28, if $P$, $Q$, and $R$ be the points where the three pairs of lines meet, the equation to the line through $P$, $Q$, and $R$ will be

$$\alpha \cos A + \beta \cos B + \gamma \cos C = 0.$$

31. The equations to the three sides of the triangle formed by joining the intersections of the angular bisectors with the opposite sides will be respectively

$$\alpha + \beta - \gamma = 0,$$
$$\beta + \gamma - \alpha = 0,$$
$$\gamma + \alpha - \beta = 0.$$

*Note.* The property of No. 11 may be applied to this theorem.

32. The three points of intersection, viz. (1) the intersection of the perpendiculars let fall from the angular points of a triangle upon the opposite sides, (2) the intersection of the lines which join the angular points with the bisections of the opposite sides, and (3) the intersection of the perpendiculars on the sides of the triangle at their middle points range in the same straight line whose equation will be

$$\alpha \sin 2A . \sin (B - C) + \beta \sin 2B . \sin (C - A) + \gamma \sin 2C . \sin (A - B) = 0.$$

33. If the three lines which join the angular points of a triangle with the center of the circumscribing circle represent in magnitude and direction three forces, the resultant force will be represented in magnitude and direction by the line which joins the center of the circumscribing circle with the intersection of the three perpendiculars let fall from the angular points on the opposite sides.

Let $ABC$ be the given triangle, $O$ the center of the circumscribing circle, the resultant of $AO$ and $BO$ will be the diagonal of the parallelogram $AGBO$, and $OG$ by construction is perpendicular to $AB$, and if $R$ be the intersection of perpendiculars let fall from the angular points upon the opposite sides, we have $OG$ evidently parallel to $CR$.

Now if $RG$ be parallel to $CO$, then $GRCO$ will be a parallelogram and $RO$ will be its diagonal, and therefore the resultant required of the forces $CO$ and $OG$, *i.e.* of $CO$, $AO$, and $BO$.

Now the equation of $RG$ can be shewn to be

$$\alpha \cos A \,.\, \sin^2 A - \beta \cos B \,.\, \sin^2 B - \gamma \cos^2 C \,.\, \sin (A - B) = 0 \ldots\ldots\ldots\ldots (1).$$

Now to decide whether $RG$ is parallel to $CO$, the question will be required to find the equation to the line through $R$ and parallel to $CO$, the resulting equation will be

$$\alpha \cos A \,.\, \sin^2 A - \beta \cos B \,.\, \sin^2 B - \gamma \cos^2 C \,.\, \sin (A - B) = 0 \ldots\ldots\ldots\ldots (2),$$

which is identical to (1) and represents $RG$.

Hence, we conclude, that $RGOC$ is a parallelogram whose diagonal is $OR$; this distance enunciated is the resultant force in magnitude and direction of three forces $AO$, $BO$, $CO$.

34. If $ABC$ be any triangle and $O$ be any point whatever in it, if through this point $O$ the lines $AD$, $BE$, and $CF$ be drawn to the opposite sides, a new triangle will be formed, viz. $DEF$. Let $DF$ be produced to meet $CA$ in $R$, $EF$ produced to meet $BC$ in $Q$, and $DE$ produced to meet $AB$ in $P$, then will $P$, $Q$, $R$ range in the same straight line. (*Vide* Remarks on this Chapter).

If we use the same notation as in question (11), we shall find the equation to the line through $P$, $Q$, $R$

$$\alpha - n\beta + nk\gamma = 0.$$

Cor. 1. If this new triangle be formed by joining the middle points of the sides of the primary triangle, the line through $P$, $Q$, $R$ will be

$$\alpha \sin A + \beta \sin B + \gamma \sin C = 0.$$

Cor. 2. If this new triangle be formed by joining the points where the feet of the perpendiculars from the angular points meet the opposite sides, the line through $P$, $Q$, $R$ will be

$$\alpha \cos A + \beta \cos B + \gamma \cos C = 0.$$

COR. 3. If the triangle be formed by joining the points where the angular bisectors meet the opposite sides, the line through $P$, $Q$, $R$ will become

$$\alpha + \beta + \gamma = 0.$$

35. The equation to the line drawn parallel to $BC$ through the center of the escribed circle which touches $BC$, is

$$(\alpha + \beta) \sin B + (\alpha + \gamma) \sin C = 0.$$

36. It will be sometimes useful to find a form for the interior angles of the three following systems of triangles:

(1) Of the triangle formed by joining the middle points of the sides of the triangle of reference.

(2) Of the triangle formed by joining the feet of the perpendiculars let fall from the angular points upon the opposite sides.

(3) Of the triangle formed by the points of intersection being joined where the angular bisectors meet the opposite sides.

In each of the foregoing cases $ABC$ is the triangle of reference and $DEF$ the new triangle formed, and having recourse to the formulæ of (17), we get

(1) which is self-evident, angle $EDF = A$,

....... $DEF = B$,

....... $EDF = C$.

(2) $\tan EDF = \tan 2A$.

(*Vide Educational Times*, Dec. 1859, p. 258).

(3) $\tan EDF = \dfrac{2(\sin B + \sin C)}{1 + 2\cos A}$.

37. Prove that the equation to the straight line passing through the middle point of the side $BC$ of a triangle, and parallel to the external bisector of the angle $A$, is

$$\beta + \gamma + \frac{a}{2}(\sin B + \sin C) = 0.$$

38. If $\alpha = 0$ and $\beta = 0$ be the equations of two straight lines, and a parallelogram be formed by drawing straight lines parallel to these at distances $a$, $b$ respectively, find the equations of the diagonals

$$\alpha b - \beta a = 0 \text{ and } \alpha b + \beta a - ab = 0.$$

39. If $\alpha = 0$, $\beta = 0$, $\gamma = 0$ be the equations to the sides of a triangle, shew that the equation to the line which joins the middle point of $AB$ with the point of intersection of the lines which bisect the angles $BAC$, $ABC$ is

$$a\alpha - b\beta - (a - b)\gamma = 0.$$

40. From the angles $ABC$ of any triangle are drawn three straight lines $AA'$, $BB'$, $CC'$ bisecting the angles, through $A$, $B$, $C$ are drawn three straight lines perpendicular to $AA'$, $BB'$, $CC'$ to meet $BC$, $CA$, and $AB$ produced in $G$, $H$, $K$; then will $G$, $H$, $K$ range in the same straight line.

The equation to that line is

$$\alpha + \beta + \gamma = 0.$$

41. In the opposite sides of a trapezium any two points $M$, $N$ are taken; the intersections of diagonals $BD$, $AC$; of the pairs $AN$, $DM$, and of $BN$, $CM$ range in the same straight line.

Let $ABCD$ be the trapezium.

Take $M$ in $AB$ and $N$ in $DC$.

Produce $AD$ and $BC$ to meet in $F$.

Join $BD$, $AC$, $AN$, $DM$, $BN$, $CM$.

Take $ABF$ as the triangle of reference.

The line required in the enunciation will be of the form

$$hm\alpha + (hm - k)\,n\beta + \{ln\,(hm - k) + h\}\,\gamma = 0,$$

$hklmn$ are arbitrary constants.

42. Let $MD$ and $BN$, also $AN$ and $MC$ be produced respectively, the line which passes through their point of intersection will pass through a fixed point.

Let $MD$ and $BN$ be produced to meet in $P'$,
$AN$ and $MC$ .............................. $Q'$.

The line through $P'Q'$ is $(k - hm)\,\alpha - hmn\beta = 0$, which evidently passes through $F$, since $ABF$ has been taken as the triangle of reference.

Therefore $F$ is a fixed point and $P'FQ'$ range in the same straight line.

43. The sides of a triangle $ABC$ are bisected forming a new triangle $A_1$, $B_1$, $C_1$ which is treated in the same way, and so on $n$ times; if $\alpha$, $\beta$, $\gamma$ be the perpendiculars from a point $(xy)$ on the sides of the triangle $ABC$, shew that the equation

$$\beta \sin B + \gamma \sin C = \frac{2^{n+1} + (-1)^n}{2^n + (-1)^{n+1}}$$

represents the line $B_nC_n$.

(1) When $n = 2$. (2) Generally.

It is quite evident from question (27) that we shall have for $B_1C_1$

$$-\alpha \sin A + \beta \sin B + \gamma \sin C = 0 \text{............} (1),$$

and if $\alpha_1\beta_1\gamma_1$ be the perpendiculars from a point $(xy)$ upon it

$$\alpha_1 = \frac{\alpha \sin A - \beta \sin B - \gamma \sin C}{2 \sin A}.$$

Similarly $$\beta_1 = \frac{-\alpha \sin A + \beta \sin B - \gamma \sin C}{2 \sin B},$$

and $$\gamma_1 = \frac{-\alpha \sin A - \beta \sin B + \gamma \sin C}{2 \sin C}.$$

Substituting these values in (1), we get, for $B_2C_2$,

$$-3\alpha \sin A + \beta \sin B + \gamma \sin C = 0 \text{.........} (2);$$

and similarly, we get

$$\alpha_2 = \frac{3\alpha \sin A - \beta \sin B - \gamma \sin C}{4 \sin A},$$

$$\beta_2 = \frac{-\alpha \sin A + 3\beta \sin B - \gamma \sin C}{4 \sin B},$$

$$\gamma_2 = \frac{-\alpha \sin A - \beta \sin B + 3\gamma \sin C}{4 \sin C};$$

therefore, for $B_3C_3$,

$$-5\alpha \sin A + 3\beta \sin B + 3\gamma \sin C = 0 \text{.........} (3).$$

By successive inferences, the coefficients of $-\alpha \sin A$ (*i. e.* numerically speaking) are of the order

$$1,\ 3,\ \frac{5}{3},\ \frac{11}{5},\ \frac{21}{11},\ \frac{43}{21},\ \&c.$$

Hence, for $B_n C_n$,

$$\beta \sin B + \gamma \sin C = \frac{2^{n+1} + (-1)^n}{2^n + (-1)^{n+1}} \sin A.$$

44. If $u + v + w - 2u' \cos A - 2v' \cos B - 2w' \cos C = 0$, then the equation

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0$$

will represent two straight lines at right angles to each other.

45. If $\alpha$, $\beta$, $\gamma$ be the trilinear coordinates of a point; $a$, $b$, $c$ the sides of the triangle of reference, and $p$, $q$, $r$ the perpendiculars from the angles on any straight line, prove that the trilinear equation of that line is

$$ap\alpha + bq\beta + cr\gamma = 0.$$

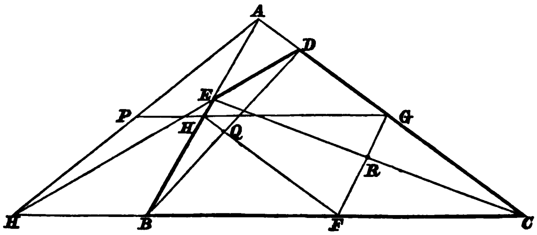
(Mr. Miller, *Educational Times*, Nov., 1861).

46. The perpendicular from any point on a right line may be represented by

$$\delta = \frac{1}{2\Delta}\,(ap\alpha + bq\beta + cr\gamma).$$

47. The middle points of the three diagonals of a complete quadrilateral range in a straight line.

Let the quadrilateral be $BCDE$. $AH$, $DB$, and $EC$ the



three diagonals. Let $ABC$ be the triangle of reference. It will be readily seen that the lines joining the middle points of the sides of the triangle $ABC$ will respectively bisect the three diagonals.

Now we can represent $DE$ by

$$l\alpha + m\beta + n\gamma = 0.$$

Hence for $AH$ $m\beta + n\gamma = 0$, and for $GH$ $-a\alpha + b\beta + c\gamma = 0$,
$DB$ $n\gamma + l\alpha = 0$, $FH$ $a\alpha - b\beta + c\gamma = 0$,
$EC$ $l\alpha + m\beta = 0$, $FG$ $a\alpha + b\beta - c\gamma = 0$.

The equation of the straight line required which contains the three points $P$, $Q$, $R$ will be

$$(bln + clm - amn)\,a\alpha + (clm + amn - bln)\,b\beta + (amn + bln - clm)\,c\gamma = 0.$$

48. From the middle points of the sides of a triangle perpendiculars are drawn (all internal or all external) and proportional to those sides; prove that the straight lines which join the angles with the extremities of the opposite perpendiculars pass through one point.

Find the condition that three points may lie in a straight line

$$\alpha_1\beta_2\gamma_3 + \alpha_2\beta_3\gamma_1 + \alpha_3\beta_1\gamma_2 = \alpha_1\beta_3\gamma_2 + \alpha_2\beta_1\gamma_3 + \alpha_3\beta_2\gamma_1.$$

49. Let the three diagonals of a complete quadrilateral be produced to meet each other in three points, and let each of these points be joined with the two opposite corners of the quadrilateral; the six lines so drawn will meet each other three and three in four points.

50. Let $ABC$ be a triangle, and $MN$ a straight line terminated by $AB$ and $AC$, and cutting them, and let $MN$ be represented by $l\alpha + m\beta + n\gamma = 0$, shew that the equation to the line joining $A$ with the middle point of $MN$ is

$$(ma - lb)\,\beta + (lc - na)\,\gamma = 0.$$

51. In the figure constructed for question 49, the four lines which meet each other in any corner of the quadrilateral are so related that two of them are parallel to the sides, and two to the diagonals of some parallelogram.

52. On the sides of the triangle $ABC$, as bases are constructed, three triangles $A'BC$, $AB'C$, $ABC'$ similar to each other, and so placed that

$$BA'C = B'AC = BAC', \quad CB'A = C'BA = CAB',$$
$$AC'B = A'CB = ACB'.$$

Prove that the straight lines $AA'$, $BB'$, $CC'$ intersect in a point.

53. If the three sides of a triangle be represented by $\alpha = 0$, $\beta = 0$, $\gamma = 0$, and the two sides $\beta = 0$ and $\gamma = 0$ be intersected by the straight line represented by $l\alpha + m\beta + n\gamma = 0$; shew that if the part of this straight line intercepted between $\beta = 0$ and $\gamma = 0$ be bisected by a straight line parallel to the side $\beta = 0$, its equation will be

$$l \sin A \,.\, \alpha + (2m \sin A - l \sin B)\, \beta + l \sin C \,.\, \gamma = 0.$$

54. The equations to the perpendicular from $(f, g, h)$ on the straight line $l\alpha + m\beta + n\gamma = 0$ are

$$\frac{\alpha - f}{l - m \cos C - n \cos B} = \frac{\beta - g}{m - n \cos A - l \cos C} = \frac{\gamma - h}{n - l \cos B - m \cos A}.$$

55. If two triangles be such that the intersections of the corresponding sides lie on the same right line, the lines joining the same, the corresponding vertices meet in a point.

56. To find the equation to a straight line perpendicular to a given straight line, and bisecting that portion of it which is terminated by the sides $AB$ and $AC$ of the triangle of reference, and from the resulting equation, shew that "If the middle points of $AB$ and $AC$ be joined, a perpendicular at the middle point of the line so formed will be

$$- \alpha \sin A \,.\, \sin (B - C) + \beta \{\sin (B - C) - 2 \sin A\}$$
$$+ \gamma \{\sin (B - C) - 2 \sin A\} = 0."$$

57. If three straight lines, drawn from the summits of a triangle, meet in one point, their respective parallels, drawn

from the middle points of the opposite sides, also meet in a point.

58. If $\alpha = 0$ and $\beta = 0$ be parallel, shew that, $k$ being an arbitrary constant, $\alpha + k\beta = 0$ is parallel to each of them.

59. Investigate what is denoted by the equation

$$\alpha \sin A + \beta \sin B + \gamma \sin C = 0.$$

60. If $D$, $E$, $F$ be the middle points of the sides of a triangle; $D'$, $E'$, $F'$ the feet of the perpendiculars let fall from the angular points, and $D''$, $E''$, $F''$ the intersections of the angular bisectors with the opposite sides; shew that perpendiculars at the middle points of $EF$, $E'F'$, and $E''F''$ will be respectively represented by

$$(1) \quad -\alpha \sin A \, . \sin (B - C) + \beta \sin B \{\sin (B - C) - 2 \sin A\} \\ + \gamma \sin C \{\sin (B - C) + 2 \sin A\} = 0.$$

$$(2) \quad \alpha \cos A \, . \sin (B - C) - \beta \sin B \, . \cos (B - C) \\ + \gamma \sin C \, . \cos (B - C) = 0.$$

$$\begin{aligned} (3) \quad & \alpha \{2 \sin B - 2 \sin C + \sin (B - C)\} \\ & - \beta \{4 \sin A + 2 \cos C \, . \sin (A - B) \\ & \qquad + 4 \sin A \, . \cos B + 2 \sin A \, . \cos C + \sin B\} \\ & - \gamma \{4 \sin A + 2 \cos B \, . \sin (A - C) \\ & \qquad + 4 \sin A \, . \cos C + 2 \sin A \, . \cos B + \sin C\} = 0. \end{aligned}$$

In a subsequent page a general solution of this property will be found.

61. Instead of employing the condition for parallelism as in question 17, Cor. 2; it will sometimes much abridge the operation by recollecting that, if one straight line be parallel to another it will only differ from it by a constant.

Thus let $ABC$ be a triangle and let the sides $AB$ and $AC$ be cut by the line $l\alpha + m\beta + n\gamma = 0$, which we will call $ED$ (*vide* fig. to question 62, p. 20).

Draw a line from $E$ parallel to the base, then that line will be represented by

$$\alpha + k(\alpha \sin A + \beta \sin B + \gamma \sin C) = 0,$$

finding out the value of $k$; when $\beta = 0$ finding the value of $\gamma$, we have, for the equation to the line required,

$$l\alpha \sin B + m\beta \sin B + m\gamma \sin C = 0.$$

Similarly, drawing a line from $D$ parallel to the base at

$$\beta = 0, \quad \gamma = -\frac{l}{n}\,\alpha,$$

and we have for its equation

$$l\alpha \sin C + n\beta \sin B + n\gamma \sin C = 0.$$

Now let it be required to draw a line through $A$ parallel to the line represented by

$$l\alpha + m\beta + n\gamma = 0.$$

Here the line will be represented by

$$l\alpha + m\beta + n\gamma + k(\alpha \sin A + \beta \sin B + \gamma \sin C) = 0,$$

or $\quad \alpha(l + k \sin A) + \beta(m + k \sin B) + \gamma(n + k \sin C) = 0,$

when the line passes through $A$

$$(l + k \sin A) = 0; \text{ therefore } k = -\frac{l}{\sin A}.$$

Hence the equation required will be

$$(m \sin A - l \sin B)\,\beta + (n \sin A - l \sin C)\,\gamma = 0.$$

Now if $MN$ be the portion of the line represented by $l\alpha + m\beta + n\gamma = 0$, terminated by the sides of the triangle, viz. $AB$ and $AC$. Then the line joining $A$ with the middle point of $MN$ will be the harmonic conjugate of the above parallel to this line; its equation will therefore be

$$(m \sin A - l \sin B)\,\beta + (l \sin C - n \sin A)\,\gamma = 0.$$

This property and its applications have been fully discussed in the *Educational Times*.

This property furnishes us with a ready method of shewing that the middle points of the three diagonals of a complete quadrilateral range in the same straight line.

Employing the notation in question (11):

In Example 34, let two co-polar triangles be *DEF*, *D′E′F′*, where *D* and *D′* are opposite to *A*, in the triangle *ABC*.

We have for

$$E'F' \quad \alpha + n'\beta + \frac{1}{m'}\gamma = 0 \text{..............} (1),$$

$$F'D' \quad \frac{1}{n'}\alpha + \beta + l'\gamma = 0 \text{..............} (2),$$

$$D'E' \quad m'\alpha + \frac{1}{l'}\beta + \gamma = 0 \text{..............} (3),$$

and for *AP*, *BQ*, *CR* respectively parallel to *EF*, *DF*, and *DE*, we have

$$AP \quad m(n \sin A - \sin B)\,\beta + (\sin A - m \sin C)\,\gamma = 0 \text{...} (4),$$

$$BQ \quad n\,(l \sin B - \sin C)\,\gamma + (\sin B - n \sin A)\,\alpha = 0 \text{...} (5),$$

$$CR \quad l\,(m \sin C - \sin A)\,\alpha + (\sin C - l \sin B)\,\beta = 0 \text{...} (6).$$

For a line passing through the intersection of *AP* and *E′F′*, or (4) and (1)

$$m'\alpha + \{m'n' + \lambda mm'(n \sin A - \sin B)\}\,\beta$$
$$+ \{1 + \lambda m'(\sin A - m \sin C)\}\,\gamma = 0 \text{......} (7),$$

where $$\lambda = -\frac{m'\alpha + m'n'\beta + \gamma}{m'\{m(n \sin A - \sin B)\beta + (\sin A - m \sin C)\gamma\}},$$

$$\beta = \frac{l(\sin A - m \sin C)}{\sin C - l \sin B},$$

$$\gamma = \frac{l(m \sin C - \sin A) - l'm'(\sin C - l \sin B)}{l'(\sin C - l \sin B)}.$$

Hence we have, for determining (7),

$$\lambda = \frac{2l}{m'\{l(m \sin C - \sin A) + ll'm(n \sin A - \sin B) + l'm'(l \sin B - \sin C)\}}.$$

Again, for a line passing through the intersection of *F′D′* and *BQ*, or (2) and (5),

$$\{1 + \lambda' n'(\sin B - n \sin A)\}\,\alpha + n'\beta$$
$$+ \{l'n' + \lambda' n'n(l \sin B - \sin C)\}\,\gamma = 0 \text{.........} (8),$$

where $$\lambda' = \frac{\alpha + n'\beta + l'n'\gamma}{n'\{n \sin C - l \sin B) + (n \sin A - \sin B)\,\alpha},$$

subject to the same conditions as in (7), viz.

$$\beta = \frac{l(\sin A - m\sin C)}{\sin C - l\sin B},$$

$$\gamma = \frac{l(m\sin C - \sin A) - l'm'(\sin C - l\sin B)}{l'(\sin C - l\sin B)},$$

whence

$$\lambda' = \frac{2l'}{lnn'(m\sin C - \sin A) + n(\sin C - l\sin B) + l'n'(n\sin A - \sin B)}.$$

Substituting the value of $\lambda$ in (7) and $\lambda'$ in (8), and recollecting that

$$lmn = l'm'n' = -1,$$

each may be reduced to the following equation of the first degree in $\alpha\beta\gamma$; and therefore is the straight line in which $P$, $Q$, and $R$ range, viz.

$$\{l(m\sin C - \sin A) + ll'm(n\sin A - \sin B) + l'm'(l\sin B - \sin C)\}m'\alpha$$
$$+\{lm'n'(m\sin C - \sin A) + lm(n\sin A - \sin B) - m'(l\sin B - \sin C)\}\beta$$
$$+\{-l(m\sin C - \sin A) + ll'm(n\sin A - \sin B) + l'm'(l\sin B - \sin C)\}\gamma = 0.$$

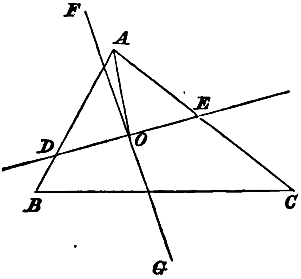
For other solutions, *vide* Ex. 74, *Mathematician*, Vol. II., p. 209. The investigation of this theorem by this method is rather long, but, it has been introduced here to exhibit the system; it sometimes happens that a geometrical solution will turn out the most concise, at other times, the application of the Cartesian method, and very often the "trilinear method"; with regard to the last mentioned application, the following question can be investigated most effectually by the application of trilinear coordinates, viz. "Prove that the angular bisector at *C* of the triangle *ABC* will meet the circumscribing circle in the point where the perpendicular at the middle point of *AB* meets the circumference."

For the investigation of this last theorem, *vide Educational Times*, June, 1859, p. 143, Cor. 1.

62. In question 56 it has been proposed to draw a line perpendicular to a given straight line at its middle point, the given line being terminated by the sides of a triangle.

In question 50, which can be readily proved (*vide Educational Times*), we have

$$\text{for } AO \quad (ma - lb)\,\beta + (lc - na)\,\gamma = 0,$$
$$DE \quad l\alpha + m\beta + n\gamma = 0\,;$$



therefore any line $FG$ can be represented by

$$l\alpha + m\beta + n\gamma + \lambda\,\{(ma - lb)\,\beta + (lc - na)\,\gamma\} = 0,$$

or $$l\alpha + \{m + \lambda\,(ma - lb)\}\,\beta + \{n + \lambda\,(lc - na)\}\,\gamma = 0 \ldots (1),$$

and in order that this line may be perpendicular to

$$l\alpha + m\beta + n\gamma = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

we must have the condition

$$\begin{aligned} &l^2 + m^2 + n^2 + \lambda m\,(ma - lb) + \lambda n\,(lc - na) \\ &= \{2mn + \lambda m\,(lc - na) + \lambda n\,(ma - lb)\}\cos A \\ &+ \{2ln + \lambda l\,(lc - na)\}\cos B \\ &+ \{2lm + \lambda l\,(ma - lb)\}\cos C \ldots\ldots\ldots\ldots\ldots\ldots (3). \end{aligned}$$

Substituting the value of $\lambda$ obtained from (2) in equation (1), we have

$$\begin{aligned} &\{(ma - lb)\,(n\cos A + l\cos C - m) \\ &\qquad + (lc - na)\,(m\cos A + l\cos B - n)\}\,l\alpha \\ &+ \{(ma - lb)\,(l^2 + n^2 - mn\cos A - 2ln\cos B - lm\cos C) \\ &\qquad + (lc - na)\,(m^2\cos A + lm\cos B - mn)\}\,\beta \\ &+ \{(ma - lb)\,(n^2\cos A + ln\cos C - mn) \\ &\qquad + (lc - na)\,(l^2 + m^2 - mn\cos A - ln\cos B - 2lm\cos C)\}\,\gamma = 0. \end{aligned}$$

Cor. 1. If it were required to find the perpendicular on the base of a triangle at its middle point; instead of equation (2), we should have

$$\alpha = 0;\ \text{and}\ l = 1,\ m = 0,\ n = 0,$$

and we have the well known equation (*vide* question 16)

$$\alpha \sin(B - C) + \beta \sin B - \gamma \sin C = 0.$$

Cor. 2. If $F$, $E$ be the middle points of $AB$ and $AC$ respectively, then

$$l = -\sin A,\quad m = \sin B,\quad n = \sin C,$$

and the equation to the perpendicular at the middle point of $EF$ will be

$$-\alpha \sin A \sin(B - C) + \beta \sin B \{\sin(B - C) - 2 \sin A\}$$
$$+ \gamma \sin C \{\sin(B - C) + 2 \sin A\} = 0.$$

Cor. 3. If $F'$, $E'$ be the feet of the perpendiculars from the angular points upon $AB$ and $BC$ respectively

$$l = -\cos A,\quad m = \cos B,\quad n = \cos C,$$

and the equation to the perpendicular through the middle point of $E'F'$ will be

$$\alpha \cos A \,.\, \sin(B - C) - \beta \sin B \,.\, \cos(B - C) + \gamma \sin C \,.\, \cos(B - C) = 0.$$

Cor. 4. If $F''$, $E''$ be the intersections of the angular bisectors with the sides $AB$ and $AC$ respectively, then

$$l = -1,\quad m = 1,\quad n = 1,$$

and we have for the perpendicular at the middle point of $E''F''$

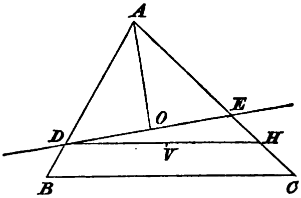
$$\alpha \{2 \sin B + 2 \sin C + \sin(B - C)\}$$
$$-\beta \{4 \sin A + 2 \cos C \,.\, \sin(A - B) + 4 \sin A \,.\, \cos B$$
$$+ 2 \sin A \,.\, \cos C + \sin B\}$$
$$+\gamma \{4 \sin A + 2 \cos B \,.\, \sin(A - C) + 4 \sin A \,.\, \cos C$$
$$+ 2 \sin A \,.\, \cos B + \sin C\} = 0.$$

63. In question 50.

Let $ADE$ be a triangle, and $MN$ a straight line terminated by $AE$, $AD$, and represented by the equation $l\alpha + m\beta + n\gamma = 0$;

shew that the equation to the line joining *A* and the middle of *MN* is $(ma - lb)\,\beta + (lc - na)\,\gamma = 0$. Also apply trilinear coordinates to prove that the middles of the three diagonals of a complete quadrilateral are in a straight line.

We can easily deduce the equation to *DH* which is parallel



to $\alpha = 0$; the middle point of which will be in the point where the line from *A* bisects *BC*.

Hence we easily determine *OV* parallel to *AC* and ultimately we have for *AO*

$$(m\alpha - l \sin B)\,\beta + (l \sin C - n \sin A)\,\gamma = 0.$$

Another method may be seen in the *Educational Times.*

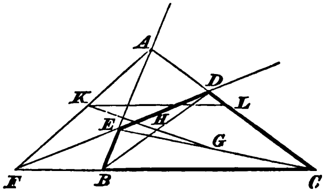
Having obtained this form we can readily proceed to prove that the points which bisect the three diagonals of a complete diagonal meet in a point.

Let *DE* be represented by

$$l\alpha + m\beta + n\gamma = 0 \text{ ....................(1);}$$

therefore for *BD* $\quad l\alpha + n\gamma = 0$ ....................(2),

*CE* $\quad l\alpha + m\beta = 0$ ....................(3);



therefore, by the last theorem, we have

$$\text{for } AG \quad (ma - lb)\,\beta + lc\gamma = 0 \;\ldots\ldots\ldots\ldots\; (4),$$

$$AH \quad lb\beta + (na - lc)\,\gamma = 0 \;\ldots\ldots\ldots\ldots\; (5),$$

and

$$AF \qquad m\beta + n\gamma = 0 \;\ldots\ldots\ldots\ldots\; (6).$$

From (3) and (4), we have

$$\lambda l\alpha + (\lambda m + ma - lb)\,\beta + lc\gamma = 0 \ldots\ldots\ldots\ldots (7),$$

and this line passes through $G$; and if we find the value of $\lambda$ by substituting the values of $\beta$ and $\gamma$, found from (1) and (6), and substitute the values thus found in (7), the equation thus found will represent the line passing through $KG$. But this line also passes through $H$, for treating the equations (2) and (5) in the same way, we obtain the same equation, viz.

$$\left.\begin{array}{l}(-\,mna + lnb + lmc)\;a\alpha \\ +\,(\;\;mna - lnb + lmc)\;b\beta \\ +\,(\;\;mna + lnb - lmc)\;c\gamma\end{array}\right\} = 0.$$

We shall conclude this Chapter with a few examples on determinants, the discussion of which will be found fully developed in Mr. Todhunter's *Theory of Equations;* also information may be derived on the same interesting subject in the *Messenger of Mathematics*, Vol. I.

### EXAMPLES.

(1). Shew that

$$\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} = \begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix}.$$

(2). Shew that

$$\begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix} = \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}.$$

(3). If there are $n$ lines and columns, shew that

$$\begin{vmatrix} a_1, & a_2, & a_3 \ldots a_n \\ b_1, & b_2, & b_3 \ldots b_n \\ c_1, & c_2, & c_3 \ldots c_n \\ \vdots & \vdots & \vdots \quad \vdots \\ k_1, & k_2, & k_3 \ldots k_n \end{vmatrix} = \begin{vmatrix} a_1, & b_1, & c_1 \ldots k_1 \\ a_2, & b_2, & c_2 \ldots k_2 \\ a_3, & b_3, & c_3 \ldots k_3 \\ \vdots & \vdots & \vdots \quad \vdots \\ k_1, & k_2, & k_3 \ldots k_n \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2, & b_3 \ldots b_n \\ c_2, & c_3 \ldots c_n \\ \vdots & \vdots \quad \vdots \\ k_2, & k_3 \ldots k_n \end{vmatrix} - b_1 \begin{vmatrix} a_2, & a_3 \ldots a_n \\ c_2, & c_3 \ldots c_n \\ \vdots & \vdots \quad \vdots \\ k_2, & k_3 \ldots k_n \end{vmatrix}$$

$$+ c_1 \begin{vmatrix} a_2, & a_3 \ldots a_n \\ b_2, & b_3 \ldots b_n \\ \vdots & \vdots \quad \vdots \\ k_2, & k_3 \ldots k_n \end{vmatrix} + \ldots + (-1)^{n-1} k_1 \begin{vmatrix} a_2, & a_3 \ldots \\ b_2, & b_3 \ldots \\ \vdots & \vdots \\ \ldots\ldots & \ldots \end{vmatrix}.$$

(4). Shew that

$$\begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1.$$

(5). Shew that

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} = \begin{vmatrix} b_2, & a_2, & c_2 \\ b_1, & a_1, & c_1 \\ b_3, & a_3, & c_3 \end{vmatrix}.$$

(6). Shew that

$$\begin{vmatrix} a_1, & b_1, & b_1 \\ a_2, & b_2, & b_2 \\ a_3, & b_3, & b_3 \end{vmatrix} = 0, \qquad \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_2, & b_2, & c_2 \end{vmatrix} = 0,$$

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = abc - af^2 - bg^2 - ch^2 + 2fgh,$$

$$\begin{vmatrix} 1, & x_1, & y_1 \\ 1, & x_2, & y_2 \\ 1, & x_3, & y_3 \end{vmatrix} = x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3.$$

(7). Shew that

$$\begin{vmatrix} 1, & \gamma, & -\beta \\ -\gamma, & 1, & \alpha \\ \beta, & -\alpha, & 1 \end{vmatrix} = 1 + \alpha^2 + \beta^2 + \gamma^2.$$

(8). Given the three equations

$$\begin{aligned} l\alpha + n'\beta + m'\gamma &= 0, \\ n'\alpha + m\beta + l'\gamma &= 0, \\ m'\alpha + l'\beta + n\gamma &= 0; \end{aligned}$$

find the determinant and shew therefrom that

$$lmn + 2l'm'n' - ll'^2 - mm'^2 - nn'^2 = 0.$$

(9). Shew that

$$\begin{vmatrix} a, & b, & c, & d \\ b, & a, & d, & c \\ c, & d, & a, & b \\ d, & c, & b, & a \end{vmatrix} = (a+b+c+d)(a-b+c-d)(a-b-c+d)(a+b-c-d).$$

(10). Eliminate $f, g, h, k$ from the equations

$$\begin{aligned} uf + w'g + v'h + lk &= 0, \\ w'f + vg + u'h + mk &= 0, \\ v'f + u'g + wh + nk &= 0, \end{aligned}$$

and

$$lf + mg + nh = 0.$$

(11). If $\begin{vmatrix} b, & c \\ y, & z \end{vmatrix} = A, \quad \begin{vmatrix} c, & a \\ z, & x \end{vmatrix} = B, \quad \begin{vmatrix} a, & b \\ x, & y \end{vmatrix} = C,$

and $\begin{vmatrix} b', & c' \\ y, & z \end{vmatrix} = A', \quad \begin{vmatrix} c', & a' \\ z, & x \end{vmatrix} = B', \quad \begin{vmatrix} a', & b' \\ x, & y \end{vmatrix} = C',$

prove that

$$\begin{vmatrix} B, & C \\ B', & C' \end{vmatrix}^2 + \begin{vmatrix} C, & A \\ C', & A' \end{vmatrix}^2 + \begin{vmatrix} A, & B \\ A', & B' \end{vmatrix}^2 = \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ x, & y, & z \end{vmatrix}^2 (x^2 + y^2 + z^2).$$

(12). Shew that

$$\begin{vmatrix} 0, & 1, & 1, & 1 \\ 1, & 0, & z^2, & y^2 \\ 1, & z^2, & 0, & x^2 \\ 1, & y^2, & x^2, & 0 \end{vmatrix} = (x+y+z)(x-y+z)(y-z-x)(z-x-y).$$

(13). Prove that

$$\begin{vmatrix} 0, & 1, & 1, & 1 \\ 1, & 0, & a+b, & a+c \\ 1, & b+a, & 0, & b+c \\ 1, & c+a, & c+b, & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = abc\ldots\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\ldots\right).$$

(14). Write out in full the expression

$$\begin{vmatrix} u, & w', & v, & a \\ w', & v, & u', & b \\ v', & u', & w, & c \\ a, & b, & c, & 0 \end{vmatrix} \phi(\alpha, \beta, \gamma) + \begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} (a\alpha + b\beta + c\gamma)^2 = 0.$$

(15). Write out in full

$$\frac{\bar{x}}{\begin{vmatrix} w', & v', & 1 \\ v, & u', & 1 \\ u', & w, & 1 \end{vmatrix}} = \frac{\bar{y}}{\begin{vmatrix} u', & w', & 1 \\ w, & v', & 1 \\ v', & u, & 1 \end{vmatrix}} = \frac{\bar{z}}{\begin{vmatrix} v', & u', & 1 \\ u, & w', & 1 \\ w', & v, & 1 \end{vmatrix}}.$$

(16). Write out in full

$$\begin{vmatrix} u, & w', & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} \phi(x, y, z) + \begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} (x+y+z)^2 = 0.$$

(17). Write out in full

$$\begin{vmatrix} 0, & f, & g, & h \\ f, & u, & w', & v' \\ g, & w', & v, & u' \\ h, & v', & u', & w \end{vmatrix} = 0.$$

(18). Write out in full

$$s = \frac{abc \begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix}}{\begin{vmatrix} u, & w', & v', & a \\ w', & v, & u', & b \\ v', & u', & w, & c \\ a, & b, & c, & 0 \end{vmatrix}}.$$

# CHAPTER II.

## ON THE EQUATION OF THE SECOND DEGREE REFERRED TO TRILINEAR COORDINATES.

64. THE general equation of the second degree is put under the form

$$l\alpha^2 + m\beta^2 + \gamma^2 + l'\beta\gamma + m'\gamma\alpha + n'\alpha\beta = 0.$$

65. The equation to the conic section passing through the angular points of the triangle of reference, *i. e.* the conic circumscribing a triangle is determined by the equation

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0,$$

or $$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0.$$

66. The equation to the conic section inscribed in a triangle is

$$\sqrt{(l\alpha)} + \sqrt{(m\beta)} + \sqrt{(n\gamma)} = 0,$$

or in its rational form

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma\alpha - 2lm\alpha\beta = 0.$$

67. The equations to the three escribed conics will be of the form

$$\sqrt{(-l\alpha)} + \sqrt{(m\beta)} + \sqrt{(n\gamma)} = 0 \text{ ......... } (1),$$

$$\sqrt{(l\alpha)} + \sqrt{(-m\beta)} + \sqrt{(n\gamma)} = 0 \text{ ......... } (2),$$

$$\sqrt{(l\alpha)} + \sqrt{(m\beta)} + \sqrt{(-n\gamma)} = 0 \text{ ......... } (3).$$

68. The condition that the straight line $A\alpha + B\beta + C\gamma = 0$ may be a tangent to the conic section represented by

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta$$

is
$$(Al)^{\frac{1}{2}} + (Bm)^{\frac{1}{2}} + (Cn)^{\frac{1}{2}} = 0.$$

69. The condition that the straight line may be a tangent to the conic section represented by $\sqrt{(l\alpha)} + \sqrt{(m\beta)} + \sqrt{(n\gamma)} = 0$ is

$$\frac{l}{A} + \frac{m}{B} + \frac{n}{C} = 0.$$

70. The condition that the line $A\alpha + B\beta + C\gamma = 0$ may touch the conic section determined by the equation

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0$$

is
$$\frac{A^2}{l} + \frac{B^2}{m} + \frac{C^2}{n} = 0.$$

71. The equation to the tangent of a conic section at the point $\beta + h\gamma = 0$, whose equation is $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$, is

$$(nh - m)^2\,\alpha + lm\beta + lnh^2\gamma = 0.$$

72. The normal of the curve $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$ at the point $\beta + h\gamma = 0$ may be represented by

$$(nh - m)\,\alpha + \lambda\beta + (l + \lambda)\,h\gamma = 0,$$

where $\lambda = \dfrac{hl^2m\cos A - h^3l^2n + u\{(u + hn)\,hl\cos B + lm\cos C - u^2\}}{lm + h^3ln - hl\,(u + 2m)\cos A - u^2\,(h\cos B - \cos C)}$,

and
$$hn - m = u.$$

Cor. 1. Normal at $A$ $(n\cos A - m)\,\beta - (m\cos A - n)\,\gamma = 0$.

Cor. 2. Normal at $B$ $(l\cos B - n)\,\gamma - (n\cos B - l)\,\alpha = 0$.

Cor. 3. Normal at $C$ $(m\cos C - l)\,\alpha - (l\cos C - m)\,\beta = 0$.

73. From the foregoing Corollaries prove that these three normals will intersect in a point if

$$\frac{l}{a}(m^2 - n^2) + \frac{m}{b}(n^2 - l^2) + \frac{n}{c}(l^2 - m^2) = 0.$$

Question 3, p. 56, Ferrers' *Trilinear Coordinates*, (*vide Educational Times*, February, 1863, Solution of Question 1303.

74. The equation to a tangent of the inscribed conic at the point $\beta - h\gamma = 0$ will be, where the conic is represented by $\sqrt{(l\alpha)} + \sqrt{(m\beta)} + \sqrt{(n\gamma)} = 0$,

$$lh^{\frac{1}{2}}\alpha - \{\sqrt{(mh)} + \sqrt{(n)}\}\, m^{\frac{1}{2}}\beta - \{\sqrt{(mh)} + \sqrt{(n)}\}\, n^{\frac{1}{2}}h^{\frac{1}{2}}\gamma = 0 \ldots (1).$$

Here it is quite evident that if the line represented by $\beta - h\gamma = 0$ cut the conic, it will meet the line represented by $l\alpha - \{\sqrt{(mh)} + \sqrt{(n)}\}^2\,\gamma = 0$ on the curve; hence the equation to a line passing through their point of intersection will be

$$l\alpha + \lambda\beta - [\{\sqrt{(mh)} + \sqrt{(n)}\}^2 + \lambda h]\,\gamma = 0 \;\ldots\ldots\; (2).$$

Subjecting this to the condition in question (69), we get

$$\lambda = -\frac{\sqrt{(m)}\,\{\sqrt{(mh)} + \sqrt{(n)}\}}{\sqrt{(h)}} \ldots\ldots\ldots\ldots\ldots (3),$$

by substitution of which we get the required equation.

COR. 1. The equation to the line through *A* and cutting the opposite side at the point of contact is

$$m\beta - n\gamma = 0.$$

Here $$h = \frac{n}{m},$$

substituting this in the above equation to the tangent, we get

$$l\alpha - 2m\beta - 2n\gamma = 0,$$

which is the well known equation to the tangent where the line through *A* first cuts the curve.

COR. 2. If the three tangents so constructed be produced to meet *BA*, *BC*, and *AC* produced, the three pairs of intersections will range on the straight line

$$l\alpha + m\beta + n\gamma = 0.$$

75. If (2), in question (74), be a normal at the point of contact of the tangent represented by (1) it will assume the form

$$l\alpha + \lambda\beta - [\{\sqrt{(mh)} + \sqrt{(n)}\}^2 + \lambda h]\,\gamma = 0,$$

where λ is equal to a fraction, the numerator of which is

$$l^2h^{\frac{1}{2}} + n^{\frac{1}{2}}h^{\frac{1}{2}} \{\sqrt{(mh)} + \sqrt{(n)}\}^3 - [m^{\frac{1}{2}} \{\sqrt{(mh)} + \sqrt{(n)}\}^2 \cos A$$
$$- lh^{\frac{1}{2}} \{2n^{\frac{1}{2}} + \sqrt{(mh)}\} \cos B + lm^{\frac{1}{2}} \cos C] \{\sqrt{(mh)} + \sqrt{(n)}\},$$

and denominator

$$(m^{\frac{1}{2}} - h^{\frac{3}{2}}) \{\sqrt{(mh)} + \sqrt{(n)}\} + h^{\frac{1}{2}} (mh - n) \cos A - lh^{\frac{3}{2}} \cos B + lh^{\frac{1}{2}} \cos C.$$

76. The equation of the tangent to the curve

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0,$$

at any point $\alpha + h\beta = 0$, will be

$$hl\alpha - m\beta - \sqrt{(-n)} \sqrt{(h^2l + m)} \gamma = 0.$$

77. If the equation in the last be of the form $l\alpha^2 - m\beta^2 - n\gamma^2$, the tangent at $\alpha + h\beta = 0$ will be

$$hl\alpha + m\beta - \sqrt{(n)} \sqrt{(h^2l - m)} \gamma = 0.$$

78. A normal to the curve $l\alpha^2 - m\beta^2 - n\gamma^2 = 0$, at the point $\alpha + h\beta = 0$, will be

$$\lambda\alpha + \{\lambda h + \sqrt{(h^2l - m)}\} \beta + \sqrt{(n)} \gamma = 0,$$

where

$$\lambda = \frac{m \sqrt{(n)} \cos A + hl \sqrt{(n)} \cos B + (n - m + hl \cos C) \sqrt{(h^2l - m)}}{(l + m) h + \{h \sqrt{(n)} \cos A + \sqrt{(n)} \cos B - \sqrt{(h^2l - m)}\} \sqrt{(h^2l - m)}},$$

and the normal at $P$ will be represented by

$$\{3 \sqrt{(lm)} - m - k\} \sqrt{(l)}\, \alpha + (l - k) \sqrt{(m)}\, \beta$$
$$+ \{2 \sqrt{(lm)} \cos C - l - m\} \sqrt{(n)}\, \gamma = 0.$$

79. The condition that the general equation

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0$$

should represent a circle is

$$vc^2 + wb^2 - 2u'bc = wa^2 + uc^2 - 2v'ca = wb^2 + va^2 - 2u'ab.$$

For investigations of these properties *vide* Ferrers' *Treatise*, p. 84, and the *Messenger of Mathematics*, Vol. I., p. 94.

80. A system of conics is described touching three straight lines; prove that if one of the foci move along a given straight line, the other will describe a conic about the triangle.

Hence prove that the circle, which passes through the point of intersection of three tangents to a parabola, passes also through the focus.

81. Find the equation to a conic touching $\alpha$, $\beta$, $\gamma$ at their middle points; the equation required is

$$\sqrt{(a\alpha)} + \sqrt{(b\beta)} + \sqrt{(c\gamma)} = 0.$$

82. The equation to the conic section touching the three sides of a triangle at the point where the angular bisectors cut the opposite sides will be

$$\sqrt{(\alpha)} + \sqrt{(\beta)} + \sqrt{(\gamma)} = 0.$$

83. The equation of a conic circumscribing the triangle of reference and having its semi-diameters parallel to the sides equal to $r_1$, $r_2$, $r_3$ respectively, is

$$\frac{a}{r_1^2\alpha} + \frac{b}{r_2^2\beta} + \frac{c}{r_3^2\gamma} = 0.$$

84. The equation to any circle can be obtained by adding to the equation of the circumscribing circle the equation of a straight line multiplied by a constant.

85. If $(\alpha \sin A + \beta \sin B + \gamma \sin C)(l\alpha + m\beta + n\gamma)$

$$+ \beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 0$$

be an equation to one circle, and

$(\alpha \sin A + \beta \sin B + \gamma \sin C)(l'\alpha + m'\beta + n'\gamma)$

$$+ \beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 0$$

be that of another, then the equation to their radical axis will be

$$(l - l')\alpha + (m - m')\beta + (n - n')\gamma = 0.$$

86. The equation $l\alpha^2 + m\beta^2 + n\gamma^2 = 0$ will represent a rectangular hyperbola if

$$l + m + n = 0.$$

87. Shew that $\sqrt{(l\alpha)} + \sqrt{(m\beta)} + \sqrt{(n\gamma)} = 0$ represents in general an ellipse, parabola, or hyperbola according as

$$\frac{l}{a} + \frac{m}{b} + \frac{n}{c}$$

is negative, zero, or positive; where $a, b, c$ denote the sides of the triangle formed by $\alpha = 0$, $\beta = 0$, $\gamma = 0$.

88. Shew that $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$ represents in general an ellipse, parabola, or hyperbola according as

$$l^2a^2 + m^2b^2 + n^2c^2 - 2lmab - 2mnbc - 2lnac$$

is negative, zero, or positive.

89. Find the equations to the tangents to the curve

$$\sqrt{(l\alpha)} + \sqrt{(m\beta)} + \sqrt{(n\gamma)} = 0,$$

which are parallel to the line $\gamma = 0$, and thence shew that the center of the curve is determined by

$$\frac{\alpha}{mc + nb} = \frac{\beta}{na + lc} = \frac{\gamma}{lb + ma}.$$

90. The center of the curve, whose equation is

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0,$$

is determined by

$$\frac{\beta}{m(l \sin A - m \sin B + n \sin C)} = \frac{\gamma}{n(l \sin A + m \sin B - n \sin C)}$$

$$= \frac{\alpha}{l(-l \sin A + m \sin B + n \sin C)}.$$

91. The equation to the circle circumscribing the triangle *ABC*, where the sides are $\alpha = 0$, $\beta = 0$, $\gamma = 0$, is

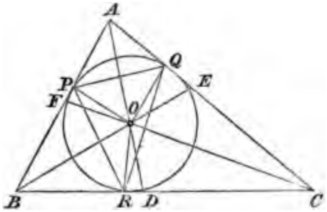
$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 0.$$

The equation to the inscribed circle is

$$\cos\frac{A}{2}\sqrt{(\alpha)}+\cos\frac{B}{2}\sqrt{(\beta)}+\cos\frac{C}{2}\sqrt{(\gamma)}=0.$$

The following will be found a good method:

Having constructed the diagram, the question will become: circumscribe a circle about the triangle *PQR*.



Let $\alpha'$, $\beta'$, $\gamma'$ be the respective perpendicular distances of $\alpha$, $\beta$, $\gamma$ from the sides of the inscribed triangle, then

$$\alpha'=\frac{\alpha\cos^{2}\frac{A}{2}-\beta\cos^{2}\frac{B}{2}-\gamma\cos^{2}\frac{C}{2}}{2\cos\frac{B}{2}\,.\cos\frac{C}{2}},$$

$$\beta'=\frac{-\alpha\cos^{2}\frac{A}{2}+\beta\cos^{2}\frac{B}{2}-\gamma\cos^{2}\frac{C}{2}}{2\cos\frac{A}{2}\,.\cos\frac{C}{2}},$$

$$\gamma'=\frac{-\alpha\cos^{2}\frac{A}{2}-\beta\cos^{2}\frac{B}{2}+\gamma\cos^{2}\frac{C}{2}}{2\cos\frac{A}{2}\,.\cos\frac{B}{2}},$$

Let now $PRQ=A'$, $PQR=B'$, $QPR=C'$.

Then $A'=\frac{\pi}{2}-\frac{A}{2}$, $B'=\frac{\pi}{2}-\frac{B}{2}$, $C'=\frac{\pi}{2}-\frac{C}{2}$.

Then the equation to the circle circumscribing *PQR* is

$$\beta'\gamma'\cos\frac{A}{2}+\gamma'\alpha'\cos\frac{B}{2}+\alpha'\beta'\cos\frac{C}{2}=0,$$

and by substitution

$$\alpha^2 \cos^4\frac{A}{2} + \beta^2 \cos^4\frac{B}{2} + \gamma^2 \cos^4\frac{C}{2}$$

$$-2\beta\gamma \cos^2\frac{B}{2}.\cos^2\frac{C}{2} - 2\alpha\gamma \cos^2\frac{A}{2}.\cos^2\frac{C}{2} - 2\alpha\beta \cos^2\frac{A}{2}.\cos^2\frac{B}{2} = 0,$$

which may obviously be reduced to the simple form

$$\cos\frac{A}{2}\sqrt{(\alpha)} + \cos\frac{B}{2}\sqrt{(\beta)} + \cos\frac{C}{2}\sqrt{(\gamma)} = 0.$$

92. The equations to the three escribed circles will be

$$\cos\frac{A}{2}\sqrt{(-\alpha)} + \sin\frac{B}{2}\sqrt{(\beta)} + \sin\frac{C}{2}\sqrt{(\gamma)} = 0,$$

$$\sin\frac{A}{2}\sqrt{(\alpha)} + \cos\frac{B}{2}\sqrt{(-\beta)} + \sin\frac{C}{2}\sqrt{(\gamma)} = 0,$$

$$\sin\frac{A}{2}\sqrt{(\alpha)} + \sin\frac{B}{2}\sqrt{(\beta)} + \cos\frac{C}{2}\sqrt{(-\gamma)} = 0.$$

93. The trilinear equation of the circle described through the three centers of the escribed circles of a triangle is

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2s(\alpha\beta + \beta\gamma + \gamma\alpha) = 0.$$

94. If the sides of a triangle, represented by $\alpha = 0$, $\beta = 0$, $\gamma = 0$, be the diameters of three circles, their equations will be

$$\alpha^2 \cos A = \beta\gamma + \gamma\alpha \cos C + \alpha\beta \cos B,$$

$$\beta^2 \cos B = \beta\gamma \cos C + \gamma\alpha + \alpha\beta \cos A,$$

$$\gamma^2 \cos C = \beta\gamma \cos B + \gamma\alpha \cos A + \alpha\beta.$$

95. The equations to the three circles which pass through the inscribed center and each of the two inscribed centers are

$$b\beta^2 + c\gamma^2 - a\alpha^2 + 2(s-a)(\beta\gamma - \gamma\alpha - \alpha\beta) = 0,$$

$$c\gamma^2 + a\alpha^2 - b\beta^2 + 2(s-b)(\gamma\alpha - \alpha\beta - \beta\gamma) = 0,$$

$$a\alpha^2 + b\beta^2 - c\gamma^2 + 2(s-c)(\alpha\beta - \beta\gamma - \gamma\alpha) = 0.$$

**96.** Find the equations, in trilinear coordinates, of the circles whose diameters are, (1) the perpendiculars drawn from the angles of a triangle to the opposite sides, (2) the lines bisecting the angles and terminated by the opposite sides, and (3) the lines joining the angles with the middles of the opposite sides.

Circle on perpendicular through $A$ is

$$(\beta \cos B - \gamma \cos C)^2 = \alpha \sin A\ (\beta \sin B + \gamma \sin C),$$

that on perpendicular through $B$

$$(\gamma \cos C - \alpha \cos A)^2 = \beta \sin B\ (\gamma \sin C + \alpha \sin A),$$

that on perpendicular through $C$

$$(\alpha \cos A - \beta \cos B)^2 = \gamma \cos C\ (\alpha \sin A + \beta \sin B).$$

In the second case, if

$AD$ be the line which bisects $BC$ in $D$,

$BE$ be the line which bisects $AC$ in $E$,

and $CF$ be the line which bisects $AB$ in $F$.

The equation to the circle described on $AD$ is

$$(\beta \sin B - \gamma \sin C)(\beta \cos B - \gamma \cos C)$$
$$= \gamma\alpha\ (\sin B + \sin C . \cos A) + \alpha\beta\ (\sin C + \sin B . \cos A).$$

That on $BE$

$$(\gamma \sin C - \alpha \sin A)(\gamma \cos C - \alpha \cos A)$$
$$= \alpha\beta\ (\sin C + \sin A . \cos B) + \beta\gamma\ (\sin A + \sin C . \cos B).$$

That on $CF$

$$(\alpha \sin A - \beta \sin B)(\alpha \cos A - \beta \cos B)$$
$$= \beta\gamma\ (\sin A + \sin B . \cos C) + \gamma\alpha\ (\sin B + \sin A . \cos C).$$

In the third case, the circle on

$AD$ is $(\beta - \gamma)(\beta \cos B - \gamma \cos C) = (1 + \cos A)(\alpha\beta + \gamma\alpha)$,

$BE$ ... $(\gamma - \alpha)(\gamma \cos C - \alpha \cos A) = (1 + \cos B)(\beta\gamma + \alpha\beta)$,

$CF$ ... $(\alpha - \beta)(\alpha \cos A - \beta \cos B) = (1 + \cos C)(\gamma\alpha + \beta\gamma)$.

97. To find the equations in trilinear coordinates of the following:

(1) Of the three circles described upon the sides of a triangle as diameters formed by joining the feet of the perpendiculars let fall from the angular points of a triangle upon the opposite sides.

(2) Those described upon the sides of the triangle formed by joining the middle points of the sides of a triangle.

(3) Those described upon the sides of the triangle formed by joining the points of intersection of the angular bisectors with the opposite sides.

The equations for (1) are

$$-\tfrac{1}{2}\alpha^2 \sin 2A . \sin B - \beta^2 \cos^3 B - \tfrac{1}{2}\gamma^2 \sin 2C . \sin B$$
$$+ \alpha\beta \cos B . \cos (A - B) + \beta\gamma \cos B . \cos (B - C) + \gamma\alpha \cos^2 B = 0$$

$$-\tfrac{1}{2}\alpha^2 \sin 2A . \sin C - \tfrac{1}{2}\beta^2 \sin 2B . \sin C - \gamma^2 \cos^3 C$$
$$+ \alpha\beta \cos^2 C + \beta\gamma \cos C . \cos (B - C) + \gamma\alpha \cos C . \cos (C - A) = 0$$

$$-\alpha^2 \cos^3 A - \tfrac{1}{2}\beta^2 \sin 2B . \sin A - \tfrac{1}{2}\gamma^2 \sin 2C . \sin A$$
$$+ \alpha\beta \cos A . \cos (A - B) + \beta\gamma \cos^2 A + \gamma\alpha \cos A . \cos (C - A) = 0.$$

The equations for (2) are

$$\alpha^2 \sin^2 A . \cos B + \beta^2 \sin^2 B . \cos B + \gamma^2 \sin C (\sin A + \sin B . \cos C)$$
$$- 2\beta\gamma \sin A . \sin B$$
$$- 2\gamma\alpha \sin A . \sin (B - C)$$
$$- 2\alpha\beta \sin A . \sin B . \cos B = 0,$$

$$\alpha^2 \sin A (\sin B + \sin C . \cos A) + \beta^2 \sin^2 B . \cos C + \gamma^2 \sin^2 C . \cos C$$
$$- 2\beta\gamma \sin B . \sin C . \cos C$$
$$- 2\gamma\alpha \sin B . \sin C$$
$$- 2\alpha\beta \sin B . \sin (C - A) = 0,$$

$$\alpha^2 \sin^2 A . \cos B + \beta^2 \sin B (\sin C + \sin A . \cos B) + \gamma^2 \sin^2 C . \cos B$$
$$- 2\beta\gamma \sin C . \sin (A - B)$$
$$- 2\gamma\alpha \sin C . \sin A . \cos A$$
$$- 2\alpha\beta \sin C . \sin A = 0.$$

The equations for (3) are

$$\begin{aligned}
\alpha^2 (1 + \cos A) + \beta^2 \cos B + \gamma^2 (1 + \cos C)& \\
- \beta\gamma (1 + \cos B + \cos C)& \\
- \gamma\alpha (1 + \cos A - 2 \cos B + \cos C)& \\
- \alpha\beta (1 + \cos A + \cos B) = 0&,
\end{aligned}$$

$$\begin{aligned}
\alpha^2 \cos A + \beta^2 (1 + \cos B) + \gamma^2 (1 + \cos C)& \\
- \beta\gamma (1 - 2 \cos A + \cos B + \cos C)& \\
- \gamma\alpha (1 + \cos A + \cos C)& \\
- \alpha\beta (1 + \cos A + \cos B) = 0&,
\end{aligned}$$

$$\begin{aligned}
\alpha^2 (1 + \cos A) + \beta^2 (1 + \cos B) + \gamma^2 \cos C& \\
- \beta\gamma (1 + \cos B + \cos C)& \\
- \gamma\alpha (1 + \cos A + \cos C)& \\
- \alpha\beta (1 + \cos A + \cos B - 2 \cos C) = 0&.
\end{aligned}$$

98. The equation to the circle circumscribing the triangle formed by joining the three middle points of the sides of a triangle is

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C - 2\beta\gamma \sin A - 2\gamma\alpha \sin B - 2\alpha\beta \sin C = 0.$$

99. To find the equation to the circle circumscribing the triangle formed by joining the feet of the perpendiculars let fall from the angular points upon the opposite sides.

In the former equation let $\alpha = 0$, then

$$(\beta \sin B - \gamma \sin C)(\beta \cos B - \gamma \cos C) = 0.$$

Hence the same circle circumscribes both triangles; but it would be a good exercise to find the equation independently.

100. To find the equation to the circle circumscribing the triangle formed by joining the points of intersection of the angular bisectors with the opposite sides.

With the usual notation, the equation

$$l\alpha^2 + m\beta^2 + n\gamma^2 - (m + n)\beta\gamma - (l + n)\gamma\alpha - (l + m)\alpha\beta = 0,$$

and we obtain

$$\frac{m+n}{l} = -\frac{v+w+\cos B+\cos C}{u+\cos A},$$

$$\frac{l+n}{l} = -\frac{u+w+\cos A+\cos C}{u+\cos A},$$

$$\frac{l+m}{l} = -\frac{u+v+\cos A+\cos B}{u+\cos A};$$

therefore $(u+\cos A)\,\alpha^2 + (v+\cos B)\,\beta^2 + (w+\cos C)\,\gamma^2$

$$\begin{aligned}&-(v+w+\cos B+\cos C)\,\beta\gamma\\ &-(u+w+\cos A+\cos C)\,\gamma\alpha\\ &-(u+v+\cos A+\cos B)\,\alpha\beta = 0.\end{aligned}$$

The form of this result is due to Mr. Watson of Haydon Bridge.

101. Prove that

$$\alpha^2\sin 2A + \beta^2\sin 2B + \gamma^2\sin 2C \\ - 2\beta\gamma\sin A - 2\gamma\alpha\sin B - 2\alpha\beta\sin C = 0$$

represents a circle. Here we must shew that

$$l\sin^2 B - n'\sin A\,.\sin B + m\sin^2 A = 4\sin A\,.\sin B\,.\sin C,$$

$$m\sin^2 C - l'\sin B\,.\sin C + n\sin^2 B = 4\sin A\,.\sin B\,.\sin C,$$

$$n\sin^2 A - m'\sin A\,.\sin C + l\sin^2 C = 4\sin A\,.\sin B\,.\sin C,$$

where $\quad l\alpha^2 + m\beta^2 - n\gamma^2 + l'\beta\gamma + m'\gamma\alpha + n'\alpha\beta = 0$

is compared with the above. (*Vide* Question 79).

102. Making use of the usual notation and construction in the circle.

(1) For $BD$ $\quad \alpha\sin C + \gamma(\sin A + \sin B) = 0,$

(2) $\quad BD'$ $\quad \gamma\sin A + \alpha(\sin B + \sin C) = 0,$

(3) $\quad AD$ $\quad \beta\sin C + \gamma(\sin A + \sin B) = 0,$

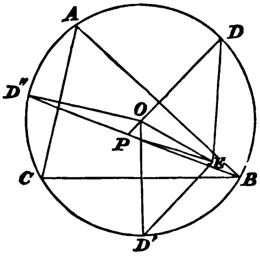
(4) $\quad D'C$ $\quad \alpha\sin B + \beta(\sin A + \sin C) = 0,$

(5) $\quad AD'$ $\quad \gamma\sin B + \beta(\sin A + \sin C) = 0,$

(6) $\quad CD'$ $\quad \beta\sin A + \alpha(\sin B + \sin C) = 0.$

Cor. 1. From (1) and (3)

$$\alpha - \beta = 0.$$



This shews that in any triangle, the bisectors of the angles and the perpendicular bisectors of the sides meet in the circumference of the circumscribing circle.

Cor. 2. Adding (1) and (4), (2) and (5), (3) and (6) in each case, we get

$$\alpha (\sin B + \sin C) + \beta (\sin A + \sin C) + \gamma (\sin A + \sin B) = 0,$$

*i.e.* the opposite sides of the hexagon inscribed in a circle meet respectively in three points, all which range in a straight line, a particular case of Pascal's theorem.

Cor. 3. If the triangle be equilateral, this line will be at infinity.

If $DE$ be parallel to $OD'$, we shall have

for $DE$ $\alpha \sin A \{\cos B + \cos (B - C)\} + \beta \sin B \{\cos A - \cos B\}$
$$+ \gamma \{\sin A + \sin B\} (1 + \cos C) = 0 \text{ ......... } (1).$$

If $D'E$ be parallel to $PD$, we shall have

for $D'E$ $\alpha (\sin B + \sin C) (1 + \cos A) + \beta \sin B (\cos C - \cos B)$
$$+ \gamma \sin C \{\cos B + \cos (A - B)\} = 0 \text{ ......... } (2).$$

The equation to the line through $P$ parallel to $D''O$ is, viz.

for $PE$ $\alpha (1 + \cos A) + \beta (\cos C - \cos A)$
$$- \gamma (1 + \cos C) = 0 \text{ ........... } (3).$$

The equation to the line through the point $O$ parallel to $D'P$ is, viz.

$$\text{for } OE \quad \alpha \sin A\{\cos C+\cos(B-C)\}+\beta \sin B(\cos C-\cos A)$$
$$-\gamma \sin C\{\cos A+\cos(A-B)\}=0 \ldots\ldots\ldots (4).$$

Subtracting (2) from (1) and a little reduction, we get

$$\alpha[\sin A\{\cos C+\cos(B-C)\}-2\sin B(1+\cos C)]$$
$$+\beta \sin B(\cos A-\cos C)$$
$$-\gamma[\sin C\{\cos B+\cos(A-B)\}-2\sin B(1+\cos A)]=0\ldots(5).$$

Hence we see that if we multiply (3) by $2\sin B$ and subtract it from (4), we obtain the same form as in (5), which is the difference between (1) and (2), or the intersection of $DE$ and $D'E$, from which we conclude that $PE$ and $OE$ intersect in $E$; hence $PEOD''$ is a parallelogram, and $OP$ is the diagonal, which is a proof of the property in the *Lady's and Gentleman's Diary*, 1850, Question 1795, proposed by Capt. A. R. Clarke, R.E., and which may be thus enunciated:

"If the three radii drawn from the center of the circumscribing circle perpendicular to the respective sides of a plane triangle represent in magnitude and direction three forces, then the line joining the center of the circumscribing circle and that of the inscribed circle will represent in magnitude and direction their resultant force." *Vide Educational Times*, the Author's contribution, June, 1859.

103. In the last figure a tangent at $D$ is

$$\sin C\{\alpha \sin A+\beta \sin B\}+\gamma(\sin A+\sin B)^2=0,$$

tangent at $D'$

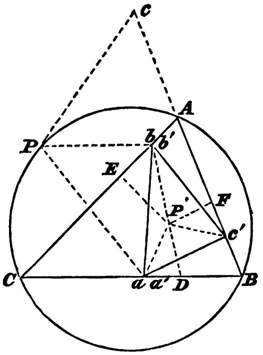
$$\sin A\{\beta \sin B+\gamma \sin C\}+\alpha(\sin B+\sin C)^2=0,$$

tangent at $D''$

$$\sin B\{\gamma \sin C+\alpha \sin A\}+\beta(\sin A+\sin C)^2=0.$$

**104. If from any point in a circle circumscribing a triangle, perpendiculars be let fall on the sides, the feet of those perpendiculars will lie in a straight line.**

**105. If from any point in a circle circumscribing a triangle, lines be drawn to the three sides of a triangle making equal angles with them, the points of intersection will lie in the same straight line.**



In the diagram, let $P'$ be any point within the triangle, and let

$$\angle P'c'A = \angle P'b'C = \angle P'a'B = \theta,$$

and if $P'F$, $P'E$, $P'D$ be drawn respectively perpendicular to the sides of the triangle $ABC$, we shall have

$$P'c' = \frac{\gamma}{\sin\theta}; \quad P'b' = \frac{\beta}{\sin\theta}; \quad P'a' = \frac{\alpha}{\sin\theta};$$

therefore

$$\text{area } a'b'c' = \frac{1}{2\sin^2\theta}\{\beta\gamma \sin b'P'c' + \gamma\alpha \sin a'P'c' + \alpha\beta \sin a'P'b'\}$$

$$= \frac{1}{2\sin^2\theta}\{\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C\}.$$

If $P'$ be removed to the circumference at $P$

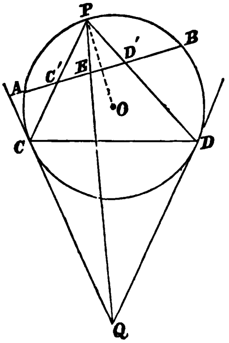
$$\text{area } (abc) = 0, \text{ since } \beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 0,$$

and $a$, $b$, $c$ range in a straight line.

106. **From the middle point $P$ of a circular arc $APB$ draw any two chords $PC$, $PD$ meeting the circumference in $C$ and $D$, and meeting the chord $AB$ in $C'$ and $D'$. Prove that the straight line joining $P$ to the pole of $CD$ will bisect $C'D'$.**

Let the equation to the line $AB$ be

$$l\alpha + m\beta + n\gamma = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (1).$$



Now if $O$ be the center of the circle, $OP$ by question must be perpendicular to $AB$; and since the equation to $OP$ will be represented by

$$\beta \cos C - \gamma \cos B = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

the condition that (2) is perpendicular to (1) is

$$l = \frac{m \sin B - n \sin C}{\sin (B - C)} \;\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Now, from Question 50, Chap. I., we know that a line joining $P$ with the middle of $C'D'$ will be determined by

$$(m \sin A - l \sin B)\, \beta + (l \sin C - n \sin A)\, \gamma = 0 \ldots (4);$$

therefore $m \sin A - l \sin B = \dfrac{\sin C\,(n \sin B - m \sin C)}{\sin (B - C)}$

by substituting (3), also

$$l \sin C - n \sin A = - \frac{\sin B\,(n \sin B - m \sin C)}{\sin (B - C)}.$$

Hence the equation (4) becomes

$$\beta \sin C - \gamma \sin B = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

which is the equation to $PE$ passing through the middle point of $C'D'$.

Again, the two tangents $QD$ and $QC$ will be represented by

$$\gamma \sin A + \alpha \sin C = 0 \text{ and } \alpha \sin B + \beta \sin A = 0 \ldots (6).$$

Hence $PQ$ may be represented by

$$\gamma \sin A + \alpha \sin C + \lambda (\alpha \sin B + \beta \sin A) = 0 \ldots (7),$$

and also by

$$\beta + \lambda'\gamma = 0 \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

From (7) and (8), we get

$$\lambda' = -\frac{\sin B}{\sin C},$$

hence (8) becomes

$$\beta \sin C - \gamma \sin B = 0 \ldots\ldots\ldots\ldots (9),$$

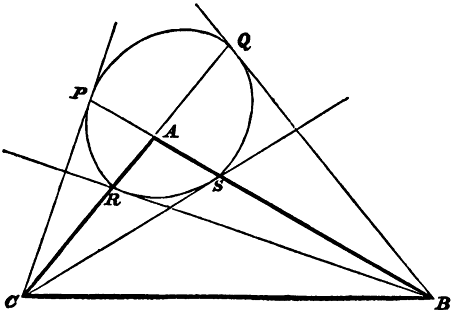
and since (9) is identical with (5) the line $PQ$ passes through $E$.

107. If the general form of the equation of the second degree be

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0$$

represent a real conic, the coefficients of $\alpha^2$, $\beta^2$, and $\gamma^2$ cannot be all of the same sign.

108. If $n$ be negative, shew that the point $A$ will be within the conic, and that if $R$ and $S$ be the points of contact where



the conic cuts $AC$ and $AB$ respectively, $P$ the point of contact of a tangent drawn from $C$; then $BC$, $CS$, $CA$, and $PC$ will form an harmonic pencil.

Here $RQ$ joins two points of contact; therefore the line

$CA$ is the polar of $B$,
$AB$ is the polar of $C$,
$C$ is the pole of $AB$,
$B$ is the pole of $AC$.

It may also be shewn that the polar $BC$ and pole $A$ are reciprocal.

109. Prove that if the equation $l\alpha^2 + m\beta^2 + n\gamma^2 = 0$ represents a parabola, then

$$\frac{a^2}{l} + \frac{b^2}{m} + \frac{c^2}{n} = 0.$$

110. If a conic section be inscribed in a quadrilateral, the lines joining the points of contact of opposite sides, each pass through the intersection of the diagonals.

Let $u = 0$, $v = 0$, $w = 0$, $t = 0$ be the equations to the sides of the quadrilateral.

If $Au + Bv + Cw = t$ with the condition $\frac{\lambda}{A} + \frac{\mu}{B} + \frac{\nu}{C} = 0$. The equation to the conic

$$(\lambda u)^{\frac{1}{2}} + (\mu v)^{\frac{1}{2}} + (\nu w)^{\frac{1}{2}} = 0.$$

Eliminating $A$, $B$, $C$ from the condition $\frac{\lambda}{A} + \frac{\mu}{B} + \frac{\nu}{C} = 0$, and from the two diagonals $Au + Bv = 0$ and $Au + Cw = 0$, we get $\lambda u - \mu v - \nu w = 0$, and this equation represents the line joining the points of contact named, which establishes the theorem.

111. The equation of the conic which touches the two sides of the triangle of reference in the points where they meet the third will be

$$k^2\alpha^2 = \beta\gamma.$$

112. The condition that the line $A\alpha + B\beta + C\gamma = 0$ may touch the conic represented by $k^2\alpha^2 = \beta\gamma$ is

$$A^2 = 4k^2BC.$$

113. The tangent will be represented at the point $\alpha - h\beta = 0$ by

$$\gamma - 2hk^2\alpha + h^2k^2\beta = 0.$$

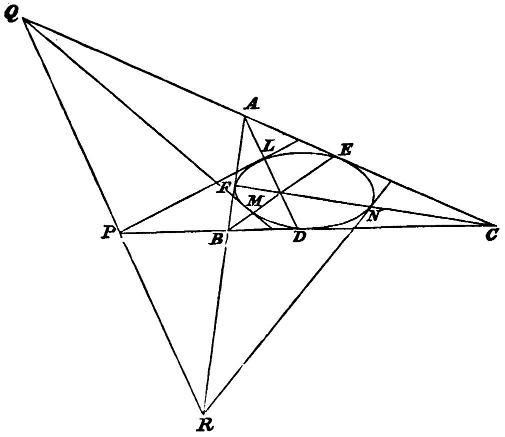
114. If the form of the equation had been $\gamma^2 = \alpha\beta$, the equation to its tangent would have been

$$h^2\alpha - 2h\gamma + \beta = 0.$$

115. If $\alpha'$, $\beta'$, $\gamma'$ be the respective values of $\alpha$, $\beta$, $\gamma$ at the pole, the equation to the polar will be

$$\alpha'\beta - 2\gamma\gamma' + \alpha\beta' = 0.$$

116. If a conic section be inscribed in a triangle, and three lines be drawn from the angular points through any point within the conic, then the three tangents at the points in the conic where the lines cut the conic, and the three sides of the



triangle produced, will meet in three points which will range in a straight line.

The equation of the conic section is

$$\sqrt{(l\alpha)} + \sqrt{(m\beta)} + \sqrt{(n\gamma)} = 0.$$

For $AD$ $m\beta - n\gamma = 0$,

.... $BE$ $n\gamma - l\alpha = 0$,

.... $CF$ $l\alpha - m\beta = 0$,

.... $LP$ $2m\beta + 2n\gamma - l\alpha = 0$,

.... $MQ$ $2n\gamma + 2l\alpha - m\beta = 0$,

.... $NR$ $2l\alpha + 2m\beta - n\gamma = 0$,

$QPR$ is a right line represented by

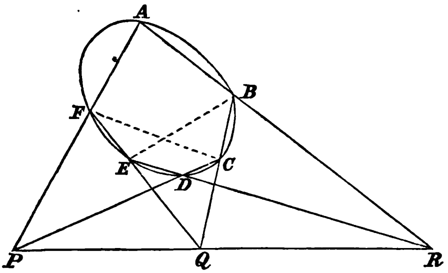
$$l\alpha + m\beta + n\gamma = 0,$$

and $CA$, $CF$, $CB$, $CR$ form an harmonic pencil.

117. *Pascal's Theorem.* The opposite pairs of sides of a hexagon inscribed in a conic section meet in three points which range in the same straight line.

For $BC$ put $u = 0$, $BE$ $P = 0$,

.... $EF$ .... $v = 0$, $CF$ $Q = 0$.



The conic passing through $BCEF$ will be represented by

$$uv + \lambda PQ = 0.$$

The equations

$$\text{for } AB \quad h'u+\lambda P=0, \quad \text{for } AF \quad v-h'Q=0,$$

$$\ldots\, DE \quad v-hP=0, \quad \ldots\, DC \quad hu+\lambda Q=0.$$

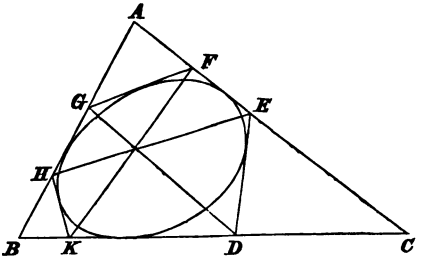
The equation to $PQR$ is

$$hh'u+\lambda v=0.$$

118. *Brianchon's Theorem.* If any hexagon be circumscribed about a conic section, the three diagonals which join its opposite angles will all pass through the same point.

Let the equation to the conic section be

$$(lu)^{\frac{1}{2}}+(mv)^{\frac{1}{2}}+(nw)^{\frac{1}{2}} \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1).$$



$$\text{For } BC \quad u=0, \quad AC \quad v=0, \quad AB \quad w=0 \;\ldots\ldots\; (2),$$

$$DE \quad \lambda u+\mu v+\nu w=0 \;\ldots\ldots\ldots\ldots\ldots\; (3),$$

$$GF \quad \lambda' u+\mu' v+\nu' w=0 \;\ldots\ldots\ldots\ldots\ldots\; (4),$$

$$GH \quad \lambda'' u+\mu'' v+\nu'' w=0 \;\ldots\ldots\ldots\ldots\ldots\; (5).$$

The condition that (3), (4), (5) may be tangents to (1), we must have

$$\frac{l}{\lambda}+\frac{m}{\mu}+\frac{n}{\nu}=1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (6),$$

$$\frac{l}{\lambda'}+\frac{m}{\mu'}+\frac{n}{\nu'}=1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (7),$$

$$\frac{l}{\lambda''}+\frac{m}{\mu''}+\frac{n}{\nu''}=1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (8).$$

The equations to the three diagonals will be, viz. for

$$GD \quad \frac{1}{\mu'\nu}\,u + \frac{1}{\lambda'\nu}\,v + \frac{1}{\lambda'\mu}\,w = 0 \ldots\ldots\ldots\ldots (9),$$

$$HE \quad \frac{1}{\mu''\nu}\,u + \frac{1}{\nu\lambda''}\,v + \frac{1}{\lambda\mu''}\,w = 0 \ldots\ldots\ldots\ldots (10),$$

$$KF \quad \frac{1}{\mu''\nu'}\,u + \frac{1}{\nu'\lambda'}\,v + \frac{1}{\lambda'\mu''}\,w = 0 \ldots\ldots\ldots\ldots (11);$$

therefore
$$\begin{vmatrix} \frac{1}{\lambda}, & \frac{1}{\mu}, & \frac{1}{\nu} \\ \frac{1}{\lambda'}, & \frac{1}{\mu'}, & \frac{1}{\nu'} \\ \frac{1}{\lambda''}, & \frac{1}{\mu''}, & \frac{1}{\nu''} \end{vmatrix} = \begin{vmatrix} \frac{1}{\mu'\nu}, & \frac{1}{\lambda'\nu}, & \frac{1}{\lambda'\mu}, \\ \frac{1}{\mu''\nu}, & \frac{1}{\lambda''\nu}, & \frac{1}{\lambda\mu''}, \\ \frac{1}{\mu''\nu'}, & \frac{1}{\lambda'\nu''}, & \frac{1}{\lambda'\mu''}, \end{vmatrix}$$

which establishes the theorem.

119. Having given a focus and two points of a conic section, prove that the locus of the intersection of the tangents at these points will be two straight lines, passing through the focus, and at right angles to each other.

120. A conic is described touching three straight lines, $BC$, $CA$, $AB$, so that the pair of tangents drawn to it from a given point $O$, are at right angles to each other. Prove that it will always touch another straight line; and that if this straight line cut $BC$, $CA$, $AB$, in $D$, $E$, $F$ respectively, each of the angles $AOD$, $BOE$, $COF$ is a right angle. And prove that the polar of $O$, with respect to this conic, will always touch a conic, of which $O$ is the focus.

121. Let the equations to the four sides, taken in order of the quadrilateral, be

$$u_1 = 0, \quad u_2 = 0, \quad u_3 = 0, \quad u_4 = 0,$$

the general type of the equation to a straight line being taken in the form

$$x \cos \alpha + y \sin \alpha - \quad = 0.$$

Then the equation to the conic section will be of the form

$$\lambda u_1 u_3 + \mu u_2 u_4 = 0,$$

when it circumscribes the quadrilateral; and if $au_1 + bu_2 + cu_3 + du_4$ vanish, for all values of $x$ and $y$,

$$\lambda b u_3 - \mu a u_4 = 0$$

will be a tangent to it.

122. Shew from the equation to the circumscribing circle about the triangle formed by three tangents to a parabola, that it passes through the focus of the parabola. *Vide* Salmon, p. 195, Ex. 3, Third Edit.

123. If the equation of a conic be expressed in terms of the equations of three right lines, the equation of the polar of the intersection of any two of them is the first derived of the equation of the conic, considered as a function of the third line.

124. Given four points on a conic, the polar of any other given point will pass through a fixed point.

The equation of the conic is of the form

$$S + kS' = 0,$$

and since $\frac{dS}{d\alpha} + k\frac{dS'}{d\alpha'} = 0$ only involves $k$ in the first degree, it will pass through a fixed point.

125. Find the locus of the pole of a given line ($\gamma$) with regard to a conic of which four points are given.

And shew also how to find the locus of the centers.

The locus of the pole is

$$\frac{dS}{d\alpha} \cdot \frac{dS'}{d\beta} - \frac{dS}{d\beta} \cdot \frac{dS'}{d\alpha} = 0.$$

126. Find the equation of the polar of any point ($\alpha'\beta'\gamma'$) with regard to a conic section.

The equation, for shortness, is

$$\alpha' \frac{dS}{d\alpha} + \beta' \frac{dS}{d\beta} + \gamma' \frac{dS}{d\gamma} = 0.$$

127. The line joining corresponding vertices of any triangle, and of its conjugate triangle with respect to a conic, meet in a point.

128. The intersections of corresponding sides of two conjugate triangles lie in one right line.

129. The anharmonic ratio of four points on a right line is the same as that of their four polars. *Vide* Salmon, p. 244, Third Edit.

130. Inscribe in a conic a triangle, the sides of which pass through three given points.

131. If two conics have a double contact, any tangent to the one is cut harmonically at its point of contact, the points where it meets the other, and where it meets the chord of contact. *Vide* Salmon, p. 245, Third Edit.

132. Shew that the three conics

$$\alpha\gamma - \frac{ln}{m^2}\beta^2 = 0, \quad \alpha\beta - \frac{lm}{n^2}\gamma^2 = 0, \quad \text{and } \beta\gamma - \frac{mn}{l^2}\alpha^2 = 0$$

meet in a single point, and that of the constants $l, m, n$ are connected by the equation $\frac{\lambda}{l} + \frac{\mu}{m} + \frac{\nu}{n} = 0$, where $\lambda, \mu, \nu$ are fixed quantities; the locus of the point is a conic section circumscribing the triangle *A*, *B*, *C*, whose sides are $\alpha = 0$, $\beta = 0$, $\gamma = 0$.

133. Two variable tangents are drawn to a conic section, so that the portion of a fixed tangent, intercepted between them, subtends a right angle at a fixed point. Prove that the locus of the point of intersection of the variable tangents is a straight line.

Also if the fixed point be a focus, the locus will be the corresponding directrix.

134. Find the equation of a conic circumscribing $(\alpha, \beta, \gamma)$, and having its center at a given point $(\alpha'\beta'\gamma')$.

For a discussion of this question, *vide* Salmon, p. 246, Third Edit., and the late Prof. Hearn's *Researches*, Chap. II.

If a fourth point be given on the conic, then

$$\frac{l}{\alpha''} + \frac{m}{\beta''} + \frac{n}{\gamma''} = 0,$$

and the locus of the center of the conic circumscribing the quadrilateral will be

$$\frac{\alpha\,(b\beta + c\gamma - a\alpha)}{\alpha''} + \frac{\beta\,(a\alpha + c\gamma - b\beta)}{\beta''} + \frac{\gamma\,(a\alpha + b\beta - c\gamma)}{\gamma''} = 0,$$

since $b\beta + c\gamma - a\alpha = 0$, $a\alpha + c\gamma - b\beta = 0$, $a\alpha + b\beta - c\gamma = 0$, &c. represent lines joining the middle points of the sides of the triangle of reference, this equation represents a conic passing through the middle points of the sides of the quadrilateral.

When three points and a tangent are given, the locus of the center will be

$$\sqrt{\{A\alpha(b\beta + c\gamma - a\alpha)\}} + \sqrt{\{B\beta(a\alpha + c\gamma - b\beta)\}} + \sqrt{\{C\gamma(a\alpha + b\beta - c\gamma)\}} = 0,$$

where $A\alpha + B\beta + C\gamma = 0$ is the equation to the tangent, and subject to the condition of tangency; and is in general a curve of the fourth degree.

135. To find the equation to a conic inscribed in $(\alpha\beta\gamma)$ and having the center at a given point $(\alpha'\beta'\gamma')$.

The locus of the centers of a conic touching three lines and passing through a given point $\alpha''\beta''\gamma''$ is

$$\sqrt{\{a\alpha''\,(b\beta + c\gamma - a\alpha)\}} + \sqrt{\{b\beta''\,(a\alpha + c\gamma - b\beta)\}}$$
$$+ \sqrt{\{c\gamma''\,(a\alpha + b\beta - c\gamma)\}} = 0,$$

the equation indicating that the conic touches the lines joining the middle points of the sides of the triangle formed by the given tangents.

Let $A\alpha + B\beta + C\gamma = 0$ be a fourth line. The locus of the center will be a right line represented by

$$\frac{a\,(b\beta + c\gamma - a\alpha)}{A} + \frac{b\,(a\alpha + c\gamma - b\beta)}{B} + \frac{c\,(a\alpha + b\beta - c\gamma)}{C} = 0.$$

136. Find the equation to the conic touching the five lines $\alpha$, $\beta$, $\gamma$, $\alpha + \beta + \gamma$, and $2\alpha + \beta - \gamma$.

Now the equation to a conic touching the three lines $\alpha$, $\beta$, $\gamma$ is

$$(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} = 0,$$

and in order that it may touch *two* lines represented respectively by

$$A\alpha + B\beta + C\gamma = 0 \text{ and } A'\alpha + B'\beta + C'\gamma = 0,$$

we must have

$$\frac{l}{A} + \frac{m}{B} + \frac{n}{C} = 0 \text{ and } \frac{l}{A'} + \frac{m}{B'} + \frac{n}{C'} = 0.$$

Now, comparing the two equations with the fourth and fifth line, we have

$$A = 1, \quad B = 1, \quad C = 1,$$

$$A' = 2, \quad B' = 1, \quad C' = -1;$$

therefore $\qquad l + m + n = 0$ and $\frac{1}{2}l + m - n = 0,$

from which we obtain

$$\frac{m}{n} = 3 \text{ and } \frac{l}{n} = -4;$$

and, by substitution, we get

$$2\,(-\alpha)^{\frac{1}{2}} + (3\beta)^{\frac{1}{2}} + (\gamma)^{\frac{1}{2}} = 0,$$

which is the required equation to the conic touching the five lines.

137. If three conic sections have one chord common to all, their three other common chords will pass through the same point. *Vide* Salmon, p. 220, Art. 267.

138. The equation of the inscribed conic touching the sides of the triangle at the feet of the perpendiculars from the opposite angles will be

$$(\alpha \cos A)^{\frac{1}{2}} + (\beta \cos B)^{\frac{1}{2}} + (\gamma \cos C)^{\frac{1}{2}} = 0.$$

139. The anharmonic ratio of four points of one conic is equal to the anharmonic ratio of the four corresponding points of the other. *Vide* Salmon, p. 227, Art. 275.

140. Find the condition that a conic circumscribed about the triangle $\alpha\beta\gamma$

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

should touch another inscribed in it represented by

$$\sqrt{(L\alpha)} + \sqrt{(M\beta)} + \sqrt{(N\gamma)} = 0 \ldots\ldots\ldots\ldots\ldots (2).$$

Rationalizing (2) and adding it to (1), we get

$$L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2$$

$$+ (l - 2MN)\,\beta\gamma + (m - 2NL)\,\gamma\alpha + (n - 2LM)\,\alpha\beta = 0 \ldots. (3).$$

Now (3) is a perfect square when

$$l = 4MN, \quad m = 4LN, \quad n = 4LM.$$

From which we obtain

$$\begin{aligned} Ll + Mm + Nn &= 12\,.\,LMN \\ &= 3 \times \{64L^3M^3N^3\}^{\frac{1}{3}} \\ &= 3L^{\frac{1}{3}}M^{\frac{1}{3}}N^{\frac{1}{3}}\,.\,\{64L^2M^2N^2\}^{\frac{1}{3}} \\ &= 3L^{\frac{1}{3}}M^{\frac{1}{3}}N^{\frac{1}{3}}\,.\,\{lmn\}^{\frac{1}{3}} \\ &= 3\,(Ll)^{\frac{1}{3}}\,.\,(Mm)^{\frac{1}{3}}\,.\,(Nn)^{\frac{1}{3}}, \end{aligned}$$

and therefore the condition that (1) touches (2) will be

$$(Ll)^{\frac{1}{3}} + (Mm)^{\frac{1}{3}} + (Nn)^{\frac{1}{3}} = 0.$$

141. Find the condition that $(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} = 0$ should represent a parabola.

Now the condition that

$$a\alpha + b\beta + c\gamma = 0$$

may be a tangent to the conic represented by

$$(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}},$$

we must have

$$\frac{l}{a} + \frac{m}{b} + \frac{n}{c} = 0,$$

which is the condition required.

142. Find the conditions that the general equation of the second degree should represent a parabola, an ellipse, or a hyperbola. *Vide Messenger of Mathematics*, Vol. I., p. 94 and 95.

143. Shew that $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$ represents in general an ellipse, parabola, or hyperbola, according as

$$l^2a^2 + m^2b^2 + n^2c^2 - 2lmab - 2mnbc - 2nlca$$

is negative, zero, or positive.

144. The equation $l\alpha^2 + m\beta^2 + n\gamma^2 = 0$ will represent a rectangular hyperbola, if $l + m + n = 0$.

145. Each angular point of a triangle is joined with each of two given points; prove that the six points of intersection of the joining lines with the opposite sides of the triangle lie in a conic section; and prove, that, if the conic described about the triangle and passing through the two given points, touch the line $(l, m, n)$, the conic passing through the six points of intersection will touch the line $\left(\frac{1}{l}, \frac{1}{m}, \frac{1}{n}\right)$.

146. Find the locus of the focus of a parabola touching $\alpha\beta\gamma$. *Vide* Salmon, p. 249, Ex. 4, Third Edit.; read also Arts. 57 and 194.

147. The equation to the directrix of the foregoing parabola will be

$$bl(\alpha \cos A - \gamma \cos C) + am(\beta \cos B - \gamma \cos C) = 0;$$

hence the directrix always passes through the intersection of the perpendiculars from the angular points upon the opposite sides.

148. Find the condition that the conic (*vide* Ferrers' *Trilinear Coordinates*, p. 67)

$$l\alpha^2 + m\beta^2 + n\gamma^2 + 2l'\beta\gamma + 2m'\gamma\alpha + 2n'\alpha\beta = 0$$

may be the product of two factors of the first degree in $\alpha\beta\gamma$. The condition is

$$lmn + 2l'm'n' - ll'^2 - mm'^2 - nn'^2 = 0.$$

149. Find the condition that the conic represented by

$$l\alpha^2 + m\beta^2 + n\gamma^2 + 2l'\beta\gamma + 2m'\gamma\alpha + 2n'\alpha\beta = 0$$

may be a rectangular hyperbola.

This question may be considered thus: since in Mr. Todhunter's *Plane Coordinate Geometry* it is shewn, that, if the curve represented by the equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

be a rectangular hyperbola,

$$a + c = 0.$$

Hence, writing the given equation in absolute terms, we find that

$$l + m + n - 2l' \cos A - 2m' \cos B - 2n' \cos C = 0.$$

150. Find the condition that the equation

$$l\beta\gamma + m\alpha\beta + n\gamma\alpha$$

may represent the rectangular hyperbola. The condition is

$$l \cos A + m \cos B + n \cos C = 0.$$

151. Every rectangular hyperbola described about a given triangle passes through the point of intersection of the perpendiculars let fall from each angular point of the triangle on the opposite side.

152. If a rectangular hyperbola be so described that each angular point of a given triangle is the pole, with respect to it of the opposite side, it will pass through the centers of the four circles which touch the three sides of the triangle.

153. To find the direction of the axis of the parabola whose given equation is

$$l\alpha^2 + m\beta^2 + n\gamma^2 + 2l'\beta\gamma + 2m'\gamma\alpha + 2n'\alpha\beta = 0.$$

*Vide Messenger of Mathematics*, Vol. I., p. 95.

154. A triangle is circumscribed to a given conic; two of its vertices move on fixed right lines, to find the locus of the third.

If the two fixed lines be represented by

$$h\alpha - \beta = 0 \text{ and } k\alpha - \beta = 0.$$

The conic by

$$\gamma^2 = \alpha\beta.$$

The locus of the vertex will be

$$(h + k)^2 (\alpha\beta) = 4hk\gamma^2.$$

*Vide* Salmon, p. 229, Third Edit.

155. Find the envelope of the base of a triangle, inscribed in a conic, and whose two sides pass through fixed points.

156. Inscribe a conic section in a triangle whose sides pass through three given points.

157. If the three sides of a triangle pass through three fixed points, and two vertices move on fixed lines, the third vertex will describe a conic section.

158. The base of a triangle touches a given conic, its extremities move on two fixed tangents to the conic, and the other two sides of the triangle pass through fixed points; find the locus of the vertex.

$\alpha = 0$ and $\beta = 0$ are the fixed tangents, and $\gamma^2 = \alpha\beta$ the conic section; $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$ two fixed points. The locus of the vertex will be

$$(\alpha\beta' - \alpha'\beta)(\alpha\beta'' - \alpha''\beta) = 4(\alpha\gamma' - \alpha'\gamma)(\gamma\beta'' - \gamma''\beta).$$

159. In the last example, if the extremities of the base lie on any conic having double contact with the given conic and passing through the given points; find the locus when the two fixed points are $\mu'\mu''$

$$(\beta - \mu'\gamma)(\mu''\alpha - \gamma) = \tan^2\phi\,(\beta - \mu''\gamma)(\mu'\alpha - \gamma),$$

where the conics are

$$\gamma^2 = \alpha\beta \text{ and } \frac{\gamma^2}{\sin^2 2\phi} - \alpha\beta = 0.$$

160. Find the envelope of a line such that the sum of the squares of the perpendiculars on it from two fixed points may be constant.

161. Find the envelope if the difference of squares of perpendiculars be given.

The envelope required will be a parabola.

162. Prove that each of the straight lines

$$u = 0, \quad v = 0, \quad w = 0$$

is the polar of the point of intersection of the other two relatively to the conic section

$$l^2u^2 + m^2v^2 = n^2w^2,$$

$l$, $m$, and $n$ being any constants.

163. To find the polar relatively to a conic section

$$uv = w^2,$$

where $u$, $v$, $w$ are linear functions of $x$ and $y$ corresponding to a pole represented by the equations

$$\left\{\begin{matrix} au = w \\ bv = w \end{matrix}\right\}.$$

The equation to the required polar is

$$au - 2abw + bv = 0.$$

164. The equation to a conic section being

$$S = l^2u^2 + m^2v^2 + n^2w^2 - 2mnvw - 2nlwu - 2lmuv = 0,$$

to prove that the equation to the polar of a point where $u = u'$, $v = v'$, $w = w'$ is

$$u'\frac{dS}{du} + v'\frac{dS}{dv} + w'\frac{dS}{dw} = 0.$$

165. To find the envelope of the conic section

$$lvw + mwu + nuv = 0,$$

$u$, $v$, $w$ being given linear functions of $x$ and $y$, and $l$, $m$, $n$ being arbitrary parameters subject to the following relation:

$$(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} = 0.$$

The required envelope is the straight line whose equation is

$$\alpha u + \beta v + \gamma w = 0.$$

166. To find the envelope of the conic section

$$(lu)^{\frac{1}{2}} + (mv)^{\frac{1}{2}} + (nw)^{\frac{1}{2}} = 0,$$

$u$, $v$, $w$ being a given linear functions of $x$ and $y$, and $l$, $m$, $n$ denoting arbitrary parameters subject to the following relation:

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0.$$

The required envelope is the straight line defined by the equation

$$\alpha u + \beta v + \gamma w = 0.$$

167. To find the envelope of a right line, such that the product of the perpendiculars drawn to it from two given points may be constant.

168. If from a fixed point $O$, $OP$ be drawn to a given right line, and the angle $TPO$ be constant, the envelope of $TP$ is a parabola having $O$ for its focus.

169. The envelope of a chord of a conic which subtends a constant angle at the focus is another conic having the same focus and the same directrix.

170. Find the curve, which is touched by all the chords of a central conic section, which subtend a right angle at the center.

171. If two fixed points $A$, $B$, on a conic be joined to a variable point $P$, and the intercept made by the joining chords on a fixed line be cut in a given ratio at $M$, the envelope of $PM$ is a conic touching parallels through $A$ and $B$ to the fixed line.

172. The six vertices of two triangles about a conic section lie also in a conic section. *Mathematician*, Vol. I., p. 326.

If the sides of one triangle be denoted by

$$u_1 = 0, \quad u_2 = 0, \quad u_3 = 0,$$

and of the other

$$v_1 = 0, \quad v_2 = 0, \quad v_3 = 0,$$

where $u_1v_1$, $u_2v_2$, $u_3v_3$ apply to the opposite sides of the hexagon, then will the equation to the conic section circumscribing both triangles be represented by

$$l(1 + h)\, u_1u_2 + k(1 + l)\, u_1u_3 + h(1 + k)\, u_2u_3,$$

$h$, $k$, $l$ are three determinable constants, and the conditions that the three diagonals pass through the origin are

$$u_1 - v_2 = h(v_1 - u_2), \quad u_2 - v_3 = k(v_2 - u_3), \quad u_3 - v_1 = l(v_3 - u_1).$$

173. If in a hexagon described about a conic section, the first, third, and fifth sides be produced till they form a triangle, and in like manner, the second, fourth, and sixth sides till they form another triangle, then the lines joining the opposite pairs of the angular points of these triangles intersect upon the three diagonals of the hexagon. *Mathematician*, Vol. II., p. 53.

This is an immediate consequence of the formula in the preceding question, using the same notation

$$u_2 + ku_3 = 0$$

is the equation to the line through the intersection of $u_2$ and $u_3$; but this equation is identical with

$$v_3 + kv_2 = 0,$$

which is a straight line through $v_2$ and $v_3$; therefore, either of these two equations represents the line joining the intersections of $u_2u_3$ with that of $r_2v_3$; and the same argument holds good with that of $u_3 + lu_1 = 0$ and $v_1 + lv_3 = 0$, and by combining these equations and reduction, we finally get

$$v_1 - u_2 = 0 \text{ or } u_1 - v_2 = 0,$$

either of which represents the diagonal of the hexagon joining $v_1u_2$, $u_1v_2$, and consequently the lines joining the intersections of

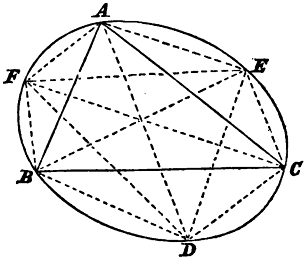
$$u_2u_3,\ v_2v_3;\quad u_1u_3,\ v_1v_3;\ v_1u_2,\ v_2u_1$$

intersect in a point.

174. If the opposite sides and diagonals of a quadrilateral inscribed in a conic section meet in $A$, $B$, $C$, and any tangent to the conic meet the sides of the triangle $ABC$ in $A'$, $B'$, $C'$; and the lines joining $A$, $B$, $C$ respectively with the intersections of $BB'$, $CC'$; $CC'$, $AA'$; $AA'$, $BB'$ meet the opposite sides in $D$, $E$, $F$; prove that $DE$, $EF$, $FD$ will also touch the conic.

175. Prove that if a conic be described about any triangle, and points where the lines bisecting the angles of the triangle meet the conic be joined, the intersections of the sides of the triangle so formed with the sides of the original triangle lie in a straight line.

$$\begin{array}{ll}
AE & l\beta + (m+n)\gamma = 0, \\
CE & m\alpha + (l+n)\beta = 0, \\
DC & (m+n)\alpha + l\beta = 0, \\
BD & l\gamma + (m+n)\alpha = 0, \\
BF & (l+m)\gamma + n\alpha = 0, \\
AF & (l+m)\gamma + n\beta = 0, \\
DE & (m+n)\alpha + (l+n)\beta - n\gamma = 0, \\
DF & (m+n)\alpha - m\beta + (l+m)\gamma = 0, \\
EF & -l\alpha + (l+n)\beta + (l+m)\gamma = 0.
\end{array}$$



The intersections of the sides of the triangles so formed lie on the straight line whose equation is

$$(m+n)\,\alpha + (l+n)\,\beta + (l+m)\,\gamma = 0.$$

*Note.* If the angular points be joined we have a particular case of Pascal's Theorem, the opposite pairs of sides of which being produced meet in three points respectively, which lie on the line represented by

$$(m+n)\,\alpha + (l+n)\,\beta + (l+m)\,\gamma = 0.$$

COR. Hence it is seen that the six points lie on the same straight line.

176. Find the equation of the chord of intersection of the circle

$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 0$$

with the curve

$$l\alpha^2 + m\beta^2 + n\gamma^2 + 2l'\beta\gamma + 2m'\gamma\alpha + 2n'\alpha\beta = 0,$$

when this latter represents a circle (*Messenger of Mathematics*, Vol. II., p. 42).

The equation to the chord required is

$$\frac{l\alpha}{a} + \frac{m\beta}{b} + \frac{n\gamma}{c} = 0.$$

*Messenger of Mathematics*, Vol. II., p. 43.

177. The equation to the *director* of the conic represented by

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma\alpha - 2lm\alpha\beta = 0$$

is $$\begin{aligned} l\cos A\alpha^2 + m\cos B\beta^2 + n\cos C\gamma^2 &- (l + m\cos C + n\cos B)\,\beta\gamma \\ &- (m + l\cos C + n\cos A)\,\gamma\alpha \\ &- (n + m\cos A + l\cos B)\,\alpha\beta = 0. \end{aligned}$$

A conic always touches four straight lines; prove that the chord of intersection of the circle described round the triangle formed by any three of these straight lines, with the director of the conic, always passes through a fixed point.

If the equation to the conic be

$$(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} = 0,$$

and $p\alpha + q\beta + r\gamma = 0$ represent the fourth line. The fixed point will be represented by

$$\frac{\alpha}{qr \cot B . \cot C} = \frac{\beta}{pr \cot A . \cot C} = \frac{\gamma}{pq \cot A . \cot B}.$$

178. From a point $A$ two chords are drawn meeting a conic section in four points $B$, joined also by four straight lines $a$. These intersect two and two, in two points $P$ lying on the polar of $A$. At the points $B$ are drawn four tangents $b$, which intersect in six points, two of which are on the polar of $A$, and the others lie two and two, on the two straight lines $AP$. These tangents intersect the original chords in four points, which may be joined by four straight lines intersecting by pairs in the points $P$. The lines $a$ and $b$ intersect in eight points $C$, which may be joined by twenty lines $c$; four of these pass through $A$, and the others may be divided into groups of four. Each group has six intersections, two of which lie on the polar of $A$, and the others lie two and two on lines through $A$. Any two groups intersect in eight points, having properties like those of the points $C$.

The above is question 1399, *Educational Times*, a solution of which was given in the August Number of 1863, by the proposer, Mr. W. K. Clifford.

179. If $\alpha\beta\gamma$ and $\alpha_1\beta_1\gamma_1$ be the trilinear coordinates of the points of intersection of the lines, joining the vertices of a triangle, with the points of contact of the inscribed and escribed circles respectively; then

$$r\sqrt{(r)}\,\{\sqrt{(\beta\gamma,\ \beta_1\gamma_1)} + \sqrt{(\gamma\alpha,\ \gamma_1\alpha_1)} + \sqrt{(\alpha\beta,\ \alpha_1\beta_1)}\}$$
$$= \sqrt{(\alpha\beta\gamma,\ \alpha_1\beta_1\gamma_1)} \cdot \sqrt{(r_1 + r_2 + r_3)}.$$

Proposed in the *Ladies and Gentleman's Diary* for 1863.

180. In a given triangle let *three* triangles be inscribed, by joining the points of contact of the inscribed circle, the

points where the bisectors of the angles meet the sides, and the points where the perpendiculars meet the sides; then will the *corresponding* sides of these triangles pass through the same point; also the triangle formed by the three points of intersection will be a *circumscribed* co-polar to the original triangle, and the pole will be on the *straight line* in which the sides of the given triangle meet the bisectors of its exterior angles.

The above is question 1273, *Educational Times*, proposed by W. J. Miller, Esq., B.A., a solution of which is given in the June Number, 1864, by Prof. Cayley.

181. If two consecutive sides of a Hexagon inscribed in a conic section are respectively parallel to the sides opposite to them, the other two sides of the Hexagon will also be parallel to each other.

182. If three sides of a quadrilateral, inscribed in a given conic section, pass always through three given points of a given straight line, to prove that the fourth side also will always pass through a given point of the same line.

183. For every point $A$ in a conic section there exists a straight line $BC$, not meeting the curve, such that if through any other point $K$ on the conic there be drawn any two straight lines meeting $BC$ in $B$, $C$, and the curve in $D$, $E$, the angles $BAC$, $DAE$ are either equal or supplementary.

184. If a conic section be described about any triangle $ABC$, and the points where the lines bisecting the angles of the triangle meet the conic be joined; the intersection of the sides of the triangle so formed with the corresponding sides of the original triangle lie in a straight line.

185. Prove that the center of the conic

$$\frac{1}{a\alpha}+\frac{1}{b\beta}+\frac{1}{c\gamma}=0$$

coincides with the center of gravity of the triangle of reference.

186. If *ABC*, *A′B′C′* be two triangles, each of which is self-conjugate with regard to the same given conic, shew that another conic can be described about both.

187. If *P* be any point on the circumference of a circle, *O* any fixed point, prove that the locus of the point, in which the tangent at *P* intersects the line which bisects *OP* at right angles, is a straight line.

188. *A*, *B*, *C*, *S* are four fixed points, *SD* is drawn perpendicular to *SA*, intersecting *BC* in *D*; *SE* perpendicular to *SB* intersecting *CA* in *E*; *SF* perpendicular to *SC* intersecting *AB* in *F*; prove that *DEF* lie in the same straight line.

189. A rectangular hyperbola circumscribes a triangle; shew that the loci of the poles of its sides are three straight lines forming another triangle, whose angular points lie on the sides of the first, where they are met by perpendiculars from the opposite angular points.

190. Prove also that the four conics which have *S* as a focus and which touch the three sides of the several triangles *ABC*, *AEF*, *BFD*, *CDE* have their latera-recta equal.

191. If, through a fixed point *O*, a straight line be drawn cutting the sides *AB*, *AC* of a triangle *ABC* in *P* and *Q* respectively, and *BC*, *CP* be joined, prove that the locus of their point of intersection is a conic circumscribing the triangle *ABC*.

192. Prove that, if the conic $(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} = 0$ be a parabola, its focus and directrix are given by the equations

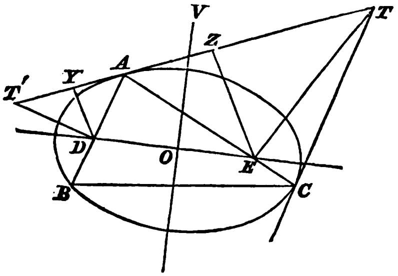
$$\frac{l\alpha}{a^2} = \frac{m\beta}{b^2} = \frac{n\gamma}{c^2},$$

$$\frac{l\alpha}{\tan A} + \frac{m\beta}{\tan B} + \frac{n\gamma}{\tan C} = 0,$$

and shew also that, if a parabola touch three straight lines, its directrix always passes through a fixed point.

193. If $PQ$ be the chord of a conic, having its pole on the chord $AB$ or $AB$ produced; and $Qq$ be drawn parallel to $AB$ meeting the conic in $q$; shew that $Pq$ bisects the chord $AB$.

194. Let an ellipse be described about a triangle $ABC$ in such a manner that the two foci may be upon the two sides of the triangle, viz. $AB$ and $AC$.



If and
$$\left.\begin{aligned} A\alpha + B\beta + C\gamma &= 0 \\ A'\alpha + B'\beta + C'\gamma &= 0 \end{aligned}\right\}$$
be two straight lines, their inclination will be determined by the equation, the numerator of which is

$$\cot\theta = AA' + BB' + CC' - (BC' + B'C)\cos A$$
$$- (AC' + A'C)\cos B - (AB' + A'B)\cos C,$$

and the denominator

$$(BC' - B'C)\sin A + (A'C - AC')\sin B + (AB' - A'B)\sin C.$$

Now let the equation to the ellipse be

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ (1);$$

therefore tangent at

$$\begin{aligned} &A \text{ will be } n\beta + m\gamma = 0, \\ &B \ldots\ldots\ldots\ l\gamma + n\alpha = 0, \\ &C \ldots\ldots\ldots\ m\alpha + l\beta = 0. \end{aligned}$$

Hence $\cot TAC = \dfrac{n - m\cos A}{m\sin A}$, $\cot T'AB = \dfrac{m - n\cos A}{-n\sin A}$;

therefore, when the foci are on the sides of the triangle

$$n = m.$$

Since, by the well known property of the ellipse, $ET$ is perpendicular to $AC$. The equation to $ET$ will be

$$m(1-\cos A)\,\alpha + (m\cos C - l\cos A)\,\beta + (m\cos C - l)\,\gamma = 0 \ldots (2).$$

Similarly, the equation to $T'D$ will be, recollecting that $n = m$,

$$m(1-\cos A)\,\alpha + (m\cos B - l)\,\beta + (m\cos B - l\cos A)\,\gamma = 0 \ldots (3).$$

Hence we must evidently have for the axis-major of the ellipse

$$m(1-\cos A)\,\alpha + (m\cos B - l)\,\beta + (m\cos C - l)\,\gamma = 0 \ldots (4).$$

We will next proceed to find an expression for the axis-minor.

The equation to the axis-minor may be readily found by making the proper substitutions in the result of question (62).

But the operation may be somewhat abridged by the following process:

We know that the center of this conic may be determined by the equations

$$\frac{\beta}{P} = \frac{\gamma}{Q} = \frac{\alpha}{R} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

where

$$\begin{aligned} P &= m(\ \ l\sin A - m\sin B + m\sin C), \\ Q &= m(\ \ l\sin A + m\sin B - m\sin C), \\ R &= \ l(-l\sin A + m\sin B + m\sin C). \end{aligned}$$

Draw a line through the vertex of the triangle of reference perpendicular to the axis-major, which may be represented by

$$\beta + h\gamma = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

and the equation to the axis-major is

$$m(1-\cos A)\,\alpha + (m\cos B - l)\,\beta + (m\cos C - l)\,\gamma = 0 \ldots (7).$$

The condition of perpendicularity between (6) and (7) is

$$h = \frac{B}{A}, \text{ where } \begin{cases} A = m\cos C - m\cos B + l\cos A - l, \\ B = m\cos C - m\cos B - l\cos A + l. \end{cases}$$

Now any line through the center may be represented by

$$\lambda P\alpha + (Q - \lambda R)\,\beta - P\gamma = 0 \ldots\ldots\ldots\ldots (8).$$

Now if (8) be parallel to (6), we must have the condition

$$\lambda = \frac{(BQ + AP)\sin A}{BR\sin A - BP\sin B - AP\sin C},$$

and we have for the axis-minor

$$\begin{aligned}(BQ + AP)\sin A\,.\,\alpha - (AR\sin A + BQ\sin B + AQ\sin C)\,\beta \\ + (BR\sin A - BP\sin B - AP\sin C) = 0,\end{aligned}$$

hence, by substitution, we have

$$(BQ + AP)\sin A = 4lm^2(\sin B - \sin C),$$

$$\begin{aligned}&AR\sin A - BP\sin B - AP\sin C\\ &= l^3\sin^2 A\,(1 - \cos A)\\ &- l^2m\sin A\,\{\sin A\,(\cos C - \cos B) + 2\sin C\,(1 - \cos A)\}\\ &- lm^2(\sin B - \sin C)\,(3\sin B + 3\sin C\cos A + \sin B\cos A + \sin C)\\ &+ m^3(\sin^2 B - \sin^2 C)\,(\cos C - \cos B),\end{aligned}$$

$$\begin{aligned}&(BR\sin A - BP\sin B - AP\sin C)\\ &\qquad = l^3\sin^2 A\,(1 - \cos A)\\ &\qquad + l^2m\sin A\,\{\sin A\,(\cos C - \cos B) - 2\sin C\,(1 - \cos A)\}\\ &\qquad - lm^2(\sin B - \sin C)^2(1 - \cos A)\\ &\qquad - m^3(\sin^2 B - \sin^2 C)\,(\cos C - \cos B).\end{aligned}$$

Therefore if the foci of the ellipse be upon the sides of the triangle of reference as enunciated, we have, for the axis-major,

$$m(1 - \cos A)\,\alpha + (m\cos B - l)\,\beta + (m\cos C - l)\,\gamma = 0,$$

and for that of the axis-minor

$$\begin{array}{r|l} 0 = \alpha & 4lm^2(\sin B - \sin C) \\ & + l^3\sin^2 A\,(1 - \cos A) \\ -\beta & - l^2m\sin A\,\{\sin A\,(\cos C - \cos B) + 2\sin C(1 - \cos A)\} \\ & + lm^2(\sin B - \sin C)(3\sin B + 3\sin C.\cos A + \sin B.\cos A + \sin C) \\ & + m^3(\sin^2 B - \sin^2 C)\,(\cos C - \cos B) \\ & + l^3\sin^2 A\,(1 - \cos A) \\ +\gamma & + l^2m\sin A\,\{\sin A\,(\cos C - \cos B) - 2\sin C(1 - \cos A)\} \\ & - lm^2(\sin B - \sin C)^2(1 - \cos A) \\ & - m^3(\sin^2 B - \sin^2 C)\,(\cos C - \cos B). \end{array}$$

The equations to the lines passing through the foci and perpendicular to the tangent at $A$, *i.e.*

for $EZ$ $\quad m(1-\cos A)\alpha+(m\cos B+\ l)\beta+(m\cos C-\ l)\gamma=0,$

$DY$ $\quad m(1-\cos A)\alpha+(m\cos B-\ l)\beta+(m\cos C-\ l)\gamma=0;$

for $ET'$ $\quad m(1-\cos A)\alpha+(m\cos B-3l)\beta+(m\cos C-\ l)\gamma=0,$

$DT$ $\quad m(1-\cos A)\alpha+(m\cos B-\ l)\beta+(m\cos C-3l)\gamma=0,$

by subtraction $\qquad \beta-\gamma=0.$

Hence $ET'$ and $DT$ intersect one another on the normal at $A$, which is the angular bisector at that point.

It is quite evident the result of the equation to the axis-minor may be derived by substituting for the values of $l$, $m$, $n$, in the resulting equation of Ex. 62, Chap. I., p. 20, the respective coefficients of $\alpha$, $\beta$, $\gamma$ in the equation to the axis-major; that equation may be first reduced into the following form, viz.

$$\begin{aligned}
&l\{l^2\sin(B-C)+(m^2-n^2)\sin A-2lm\sin B+2ln\sin C\}\alpha\\
&+[l^2(l\sin B-2m\sin A-n\sin 2B)+m^2\{l\sin(A-C)+n\sin 2A\}\\
&\qquad\qquad+n^2(l\sin B-2m\sin A)+4lmn\sin A\,.\cos B]\beta\\
&-[l^2(l\sin C-2n\sin A-m\sin 2C)+m^2(l\sin C-2n\sin A)\\
&\qquad\qquad+n^2\{l\sin(A-B)+m\sin 2A\}+4lmn\sin A\,.\cos C]\gamma=0.
\end{aligned}$$

195. An ellipse is described so as to touch the asymptotes of an hyperbola; shew that two of the chords joining the points of intersection of the ellipse and hyperbola are parallel.

196. If $\alpha\beta=c^2$ be the equation to an hyperbola, then $\alpha\beta=0$, $\alpha^2-\beta^2=0$, $\alpha^2-n^2\beta^2=0$ are the respective equations to the asymptotes, the axes and a pair of conjugate diameters, $n$ being constant.

197. Find the locus of the intersection of a perpendicular through the focus with the tangent, and apply the condition proposed in question (79) to shew that it will be a circle, the foci of the ellipse being in the sides of the triangle.

198. Also when the foci are in the sides of the triangle, find the locus of two tangents at right angles to each other.

199. Prove that the straight lines represented by the equation

$$(\alpha^2 - \beta^2) \sin C + k(\alpha \sin A + \beta \sin B)(\beta \cos B - \alpha \cos A) = 0$$

are parallel to the axes of the conic section $2\alpha\beta = k\gamma^2$, where $A$, $B$, $C$ are the angles of the triangle formed by the three given straight lines.

200. The equations of three straight lines being $\alpha = 0$, $\beta = 0$, $\gamma = 0$, and of two conics

$$\alpha\beta = \mu^3\gamma^2 \text{ and } \gamma\alpha = \lambda^3\beta^2 \text{ .............. } (1),$$

let it be required to find a tangent common to both.

Let this tangent be represented by

$$l\alpha + m\beta + n\gamma = 0 \text{ ...................} (2);$$

therefore $$\alpha = -\frac{m\beta + n\gamma}{l} \text{ and } \gamma = -\frac{l\alpha + m\beta}{n},$$

substituting these values in (1), two secants will be represented, one to each conic; and if these secants become tangents they must be perfect squares, and we get the conditions

$$\frac{m}{n} = \frac{\lambda}{\mu}, \quad \frac{m}{l} = 4\mu\lambda^2, \text{ and } \frac{n}{l} = 4\lambda\mu^2;$$

substituting these values in (2), we get, for the equation to the tangent common to the two conics,

$$\alpha + 4\lambda\mu(\lambda\beta + \mu\gamma) = 0.$$

201. A triangle $ABC$ is inscribed in a conic, and has the interior bisector of its angle $A$, a normal to the curve; shew that the side $BC$ passes through the pole of the normal.

Since $\beta - \gamma = 0$ is the normal at $A$,

$\beta + \gamma = 0$ must be the tangent at $A$, and the tangent at the other extremity of the pole is

$$2\alpha + \beta + \gamma = 0.$$

Hence the tangent $\beta + \gamma = 0$ intersects $BC$ at the pole of $\beta - \gamma = 0$.

202. Shew that

$$\begin{vmatrix} \left(\dfrac{as\cos A}{r^2} - u\right), & -w', & -v', & a \\ -w', & \left(\dfrac{bs\cos B}{r^2} - v\right), & -u', & b \\ -v', & -u', & \left(\dfrac{cs\cos C}{r^2} - w\right), & c \\ a, & b, & c, & 0 \end{vmatrix} = 0$$

is the equation for determining the axes of the conic whose equation is

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0,$$

where
$$s = abc\,\frac{\begin{vmatrix} u, & w', & v \\ w', & v, & u' \\ v', & u', & w \end{vmatrix}}{\begin{vmatrix} u, & w', & v', & a \\ w', & v, & u', & b \\ v', & u', & w, & c \\ a, & b, & c, & 0 \end{vmatrix}}.$$

For the investigation of this question (*vide* Ferrers' *Trilinear Coordinates*, p. 89).

203. Prove that the area of the foregoing conic will be

$$2\pi\Delta abc = \frac{\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix}}{\begin{vmatrix} u, & w', & v', & -a \\ w', & v, & u', & -b \\ v', & u', & w, & -c \\ a, & b, & c, & 0 \end{vmatrix}}.$$

204. If an ellipse of given area be circumscribed about a given triangle the locus of the center referred to the same triangle will be represented by the equation

$$(b\beta + c\gamma - a\alpha)(c\gamma + a\alpha - b\beta)(a\alpha + b\beta - c\gamma) = C\alpha^2\beta^2\gamma^2,$$

$C$ being a constant depending on the lengths of the sides of the triangle (*vide Solutions of the Senate-House Problems and Riders*, by the Moderators and Examiners, for 1864, p. 164).

205. Let the conic about $ABC$ in (Ex. 194) be the hyperbola. Now considering when the hyperbola and the ellipse have the same foci, the tangents at their point of contact are at right angles to each other, we shall have, for the tangent at $A$,

$$\beta - \gamma = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now the tangents at $B$ and $C$ must be respectively represented by

$$(l\cos B - n)\gamma + (l - n\cos B)\alpha = 0 \ldots\ldots\ldots(2),$$

and
$$(m\cos C - l)\alpha + (m - l\cos C)\beta = 0 \ldots\ldots\ldots(3).$$

Hence we obtain the equations to the two perpendiculars from the foci upon the tangent, viz.

$$(1 + \cos A)(l - n\cos B)\alpha - n\sin^2 B.\beta$$
$$+ \{(l\cos B - n)(1 + \cos A) + n\sin^2 B\}\gamma = 0 \ldots\ldots(4),$$

and $(1 + \cos A)(m\cos C - l)\alpha$

$$+ \{(m - l\cos C)(1 + \cos A) - m\sin^2 C\}\beta - (1 + \cos A)\gamma = 0 \ldots(5);$$

therefore, for the equation to the axis-major, we have

$$(1 + \cos A)\alpha - \frac{n\sin^2 B}{l - n\cos B}\beta - \frac{1 + \cos A}{m\cos C - l}\gamma = 0,$$

but since the tangent at any point of a hyperbola makes equal angles with the focal distances, we have

$$n = -m,$$

and therefore the equation to this axis becomes

$$(1 + \cos A)\alpha + \frac{m\sin^2 B}{l + m\cos B}\beta + \frac{1 + \cos A}{l + m\cos C}\gamma = 0.$$

To find the axis-minor, in Question 62, Chap. I., substitute for $l, m, n$ in the resulting formula

$$1 + \cos A, \quad \frac{m \sin^2 B}{l + m \cos B}, \text{ and } \frac{1 + \cos A}{l + m \cos C}.$$

206. To find expressions for the axes of the conics represented by the equations

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0 \text{ .................. } (1),$$

and $$\sqrt{\{(l\alpha)\}} + \sqrt{\{(m\beta)\}} + \sqrt{\{(n\gamma)\}} = 0 \text{ ............ } (2).$$

In the first case let

$$P = m(\ \ l \sin A - m \sin B + n \sin C),$$
$$Q = n(\ \ l \sin A + m \sin B - n \sin C),$$
$$R = l(- l \sin A + m \sin B - n \sin C).$$

In the second let

$$P = na + lc,$$
$$Q = lb + ma,$$
$$R = mc + nb.$$

In each of these cases the centers will be determined by

$$\frac{\beta}{P} = \frac{\gamma}{Q} = \frac{\alpha}{R} \text{ ........................ } (3).$$

Now if $\lambda$ and $\lambda'$ be two arbitrary constants, and two lines passing through the center may be represented by

$$\lambda Q\alpha - Q\beta + (P - \lambda R)\gamma = 0 \text{ ............... } (4),$$
$$\lambda' P\alpha - (Q + \lambda' R)\beta + P\gamma = 0 \text{ ............... } (5).$$

Now the equation to a tangent to (1) at the point $\beta + h\gamma = 0$ is

$$(nh - m)^2 \alpha + lm\beta + h^2 ln\gamma = 0 \text{ ............ } (6),$$

and the equation to a tangent to (2) at a point $\beta - h\gamma = 0$ is

$$lh^{\frac{1}{2}}\alpha - \{\sqrt{(mh)} + \sqrt{(n)}\}\, m^{\frac{1}{2}}\beta - \{\sqrt{(mh)} + \sqrt{(n)}\}\, n^{\frac{1}{2}}h^{\frac{1}{2}}\gamma = 0 \text{ ... } (7).$$

Either (6) or (7) may be represented by

$$X\alpha + Y\beta + Z\gamma = 0 \text{ ............... } (8),$$

(4) may be written $$X'\alpha + Y'\beta + Z'\gamma = 0 \text{ ............... } (9),$$

(5) may be written $$X''\alpha + Y''\beta + Z''\gamma = 0 \text{ ............... } (10).$$

Now if (9) be the axis-minor it must be parallel to (8), and we have the condition

$$(YZ'-ZY')\sin A+(XZ'-Z'X)\sin B+(XY'-X'Y)\sin C=0,$$

whence $$(Z\sin B+Y\sin C)\frac{X'}{Y'}-(Y\sin A+X\sin B)\frac{Z'}{Y'}$$
$$-(X\sin C-Z\sin A)=0\ldots\ldots\ldots\ldots(11).$$

Now comparing (9) and (4), we get

$$\frac{X'}{Y'}=-\lambda \text{ and } \frac{Z'}{Y'}=-\frac{P-\lambda R}{Q}.$$

From which we obtain

$$\lambda=\frac{P(Y\sin A+X\sin B)-Q(X\sin C-Z\sin A)}{Q(Z\sin B+Y\sin C)+R(Y\sin A+X\sin B)},$$

$$\frac{P-R\lambda}{Q}=\frac{P(Z\sin B+Y\sin C)+R(X\sin C-Z\sin A)}{Q(Z\sin B+Y\sin C)+R(Y\sin A+X\sin B)}.$$

Therefore the equation for the axis-minor will become

$$\{P(Y\sin A+X\sin B)-Q(X\sin C-Z\sin A)\}\alpha$$
$$-\{Q(Z\sin B+Y\sin C)+R(Y\sin A+X\sin B)\}\beta$$
$$+\{P(Z\sin B+Y\sin C)+R(Y\sin A+X\sin B)\}\gamma=0\ldots(12).$$

Again, for the axis-major we must have the condition

$$XX''+YY''+ZZ''=(YZ''+Y''Z)\cos A$$
$$+(X''Z+XZ'')\cos B+(X''Y+XY'')\cos C=0;$$

therefore $$(X-Z\cos B-Y\cos C)\frac{X''}{Y''}+(Z-Y\cos A-X\cos B)\frac{Z''}{Y''}$$
$$=Z\cos A+X\cos C-Y.$$

But from comparing (5) and (10), we have

$$\frac{X''}{Y''}=-\frac{\lambda' P}{Q+\lambda' R} \text{ and } \frac{Z''}{Y''}=-\frac{P}{Q+\lambda' R}.$$

Hence we get

$$\lambda'=\frac{Q(Y-Z\cos A-X\cos C)-P(Z-Y\cos A-X\cos B)}{P(X-Z\cos B-Y\cos C)-R(Y-Z\cos A-X\cos C)},$$

and by substitution we obtain

$$\{Q(Y - Z\cos A - X\cos C) - P(Z - Y\cos A - X\cos B)\}\,\alpha$$
$$-\{Q(X - Z\cos B - Y\cos C) - R(Z - Y\cos A - X\cos B)\}\,\beta$$
$$+\{P(X - Z\cos B - Y\cos C) - R(Y - Z\cos A - X\cos C)\}\,\gamma = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

207. Let an ellipse be inscribed in a triangle $ABC$ in such a manner that the axis-major may coincide with the angular bisector at $A$.

Now we can write

$$P \text{ for } na + lc,$$
$$Q \text{ for } lb + ma,$$
$$R \text{ for } mc + nb,$$

and the center of the conic section may be determined by the equations

$$\frac{\alpha}{R} = \frac{\beta}{P} = \frac{\gamma}{Q}.$$

Then let it be required to find the axis-minor, since the expression for the axis-minor is {*vide* (12), p. 74}

$$\{P(Y\sin A + X\sin B) - Q(X\sin C - Z\sin A)\}\,\alpha$$
$$-\{Q(Z\sin B + Y\sin C) + R(Y\sin A + X\sin B)\}\,\beta$$
$$+\{P(Z\sin B + Y\sin C) + R(Y\sin A + X\sin B)\}\,\gamma = 0,$$

and the equation to a tangent at the point $\beta - h\gamma = 0$,

$$lh^{\frac{1}{2}}\alpha - \{\sqrt{(mh)} + \sqrt{(n)}\}\,m^{\frac{1}{2}}\beta - \{\sqrt{(mh)} + \sqrt{(n)}\}\,n^{\frac{1}{2}}h^{\frac{1}{2}}\gamma = 0.$$

Then, for the tangent where $\beta - h\gamma = 0$ cuts the conic, we must have $h = 1$; therefore

$$l\alpha - \{\sqrt{(m)} + \sqrt{(n)}\}\,\sqrt{(m)}\,\beta - \{\sqrt{(m)} + \sqrt{(n)}\}\,\sqrt{(n)}\,\gamma = 0.$$

In order that this may be a tangent at the extremity of the axis-major, we must have the condition that it may be parallel to the line represented by

$$\beta + \gamma = 0.$$

That condition will be

$$l = \frac{(m - n)\sin A}{\sin B + \sin C},$$

and this tangent becomes

$$\{\sqrt{(m)} - \sqrt{(n)}\} \sin A . \alpha - m^{\frac{1}{2}} (\sin B + \sin C) \beta - n^{\frac{1}{2}} (\sin B + \sin C) \gamma = 0,$$

and we have

$$X = \{\sqrt{(m)} - \sqrt{(n)}\} \sin A,$$
$$Y = - m^{\frac{1}{2}} (\sin B + \sin C),$$
$$Z = - n^{\frac{1}{2}} (\sin B + \sin C).$$

Substituting these values of $X$, $Y$, $Z$ in the expression for the axis-minor, we obtain the equation

$$(P + Q) \sin A . \alpha - \{Q (\sin B + \sin C) + R \sin A\} \beta$$
$$+ \{P (\sin B + \sin C) - R \sin A\} \gamma = 0.$$

208. Supposing it were required to find a tangent to this inscribed conic, such that it should be perpendicular to the side $AC$ of the triangle of reference where $\beta = 0$.

Let $$A\alpha + B\beta + C\gamma = 0 \text{ ..................... } (1)$$
represent that line.

Now since it is perpendicular to $\beta = 0$, we must have

$$B = A \cos C + C \cos A,$$

also combining this with the condition

$$\frac{l}{A} + \frac{m}{B} + \frac{n}{C} = 0,$$

we get

$$A^2 n \cos C + AC (n \cos A + l \cos C + m) + C^2 l \cos A = 0 \text{...(2)};$$

therefore $$A \sqrt{(n \cos C)} + C \sqrt{(l \cos A)} = 0;$$

therefore $$\frac{A}{C} = - \frac{\sqrt{(l \cos A)}}{\sqrt{(n \cos C)}};$$

therefore $$\frac{B}{C} = - \frac{\sqrt{(l \cos A . \cos C)} - \sqrt{(n)} \cos A}{\sqrt{(n)}},$$

and $$l \cos C - 2 \sqrt{(ln \cos A . \cos C)} + n \cos A + m = 0 \text{...(3)},$$

and we have, for the tangent required,

$$\sqrt{(l \cos A)} \alpha + \sqrt{(\cos A . \cos C)} \{\sqrt{(l \cos C)} - \sqrt{(n \cos A)}\} \beta$$
$$- \sqrt{(n \cos C)} \gamma = 0$$

and since from (3) we get

$$\sqrt{(l\cos C)} - \sqrt{(n\cos A)} = \pm\sqrt{(-m)},$$

and substituting the positive sign, we have, for the tangent,

$$\sqrt{(l\cos A)}\,\alpha + \sqrt{(-m\cos A\,.\cos C)}\,\beta - \sqrt{(n\cos C)}\,\gamma = 0 \ldots (4).$$

That this equation represents a tangent to the conic is manifest since

$$\frac{l}{\sqrt{(l\cos A)}} + \frac{m}{\sqrt{(-m\cos A\,.\cos C)}} - \frac{n}{\sqrt{(n\cos C)}} = 0,$$

*i.e.* $$\frac{l}{\sqrt{(\cos A)}} - \frac{-m}{\sqrt{(-m\cos A\,.\cos C)}} - \frac{n}{\sqrt{(n\cos C)}} = 0,$$

or $$\sqrt{(l\cos C)} - \sqrt{(-m)} - \sqrt{(n\cos A)} = 0,$$

which is true, since

$$\sqrt{(l\cos C)} - \sqrt{(n\cos A)} = \sqrt{(-m)}.$$

It is also perpendicular to the side of the triangle $\beta = 0$, as it satisfies the criterion, viz.

$$\cos(\alpha-\beta)\sqrt{(l\cos A)} + \sqrt{(-m\cos A.\cos C)} - \cos(\beta-\gamma)\sqrt{(n\cos C)} = 0.$$

In the same manner it may be proved that the equation to a tangent perpendicular to $AB$ of the triangle of reference, where $\gamma = 0$, will be

$$\sqrt{(l\cos A)}\,\alpha - \sqrt{(m\cos B)}\,\beta + \sqrt{(-n\cos A\,.\cos B)}\,\gamma = 0,$$

which is true, being subject to the required conditions of tangency and perpendicularity, with a further condition that

$$\sqrt{(l\cos B)} - \sqrt{(m\cos A)} = \sqrt{(-n)}.$$

209. A parabola described about a triangle is determined by the equation $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$; it is required to determine the position of the focus when it lies in the base $BC$ of the triangle of reference.

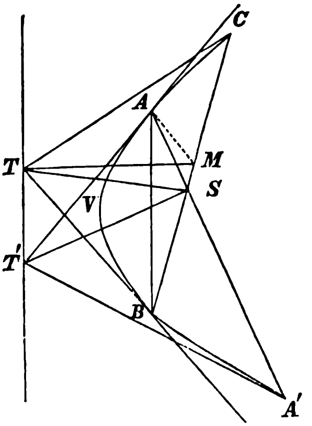
The tangent at

$A$ is $n\beta + m\gamma = 0$ .................... (1),

$B$ ... $l\gamma + n\alpha = 0$ .................... (2),

$C$ ... $m\alpha + l\beta = 0$ .................. (3).

Now if the tangents be drawn at the extremities of the focal chords they will intersect each other at right angles on the



directrix, and the lines joining the points of intersection of the tangents so drawn are perpendicular to the focal chords.

The condition that (2) and (3) may be perpendicular to each other is

$$n = \frac{l\,(l\cos A + m\cos B)}{m - l\cos C},$$

and the line $ST$ may be represented by

$$(m + \lambda n)\,\alpha + l\beta + \lambda l\gamma = 0,$$

where $$\lambda = \frac{m - l\cos C}{l\cos B - n} = -\frac{(m - l\cos C)^2}{l^2\sin B\,.\sin C};$$

therefore $$m + \lambda n = m - \frac{(m - l\cos C)\,(l\cos A + m\cos B)}{l\sin B\,.\sin C}$$

$$= \frac{l^2\cos A\,.\cos C + 2lm\cos B\,.\cos C - m^2\cos B}{l\sin B\,.\sin C};$$

therefore the equation to $ST$ is

$$(l^2 \cos A . \cos C + 2lm \cos B . \cos C - m^2 \cos B)\,\alpha$$
$$+ l^2 \sin B . \sin C . \beta - (m - l \cos C)^2\,\gamma = 0,$$

hence the position of the focus $S$ is determined.

If $\alpha = 0$, we have, for $ASA'$,

$$l^2 \sin B . \sin C . \beta - (m - l \cos C)^2\,\gamma = 0,$$

and the equation to $ST'$ will be easily determined.

210. Required to find the normals at $C$ and $B$.

A normal at $C$ will be represented by

$$(m \cos C - l)\,\alpha + (m - l \cos C)\,\beta = 0,$$

and the normal at $B$, bearing in mind that

$$n = \frac{l(l \cos A + m \cos B)}{m - l \cos C},$$

since in the parabola the two tangents are perpendicular to each other, will be

$$l \sin C . \gamma + (l \sin A - m \sin B)\,\alpha = 0.$$

Again, if $M$ be the middle point of $BC$, we shall have, for $AM$,

$$\beta \sin B - \gamma \sin C = 0,$$

and it is known that a line passing through the intersection of the normals and the middle point of $BC$ will be parallel to the axis; and since the tangents and normals here constructed form a parallelogram, of which this line must form one of the diagonals which will evidently pass through $T$, and can therefore be represented by

$$(l^2 \sin C . \cos A + lm \sin A - m^2 \sin B)\,\alpha$$
$$- l(m - l \cos C)(\beta \sin B - \gamma \sin C) = 0.$$

Now the axis of the parabola will pass through the intersection of the lines

$$\alpha = 0 \text{ and } l^2 \sin B . \sin C . \beta - (m - l \cos C)^2\,\gamma = 0,$$

and the axis will be determined by the equation

$$\{l^2 \sin C . \cos A + lm \sin A - m^2 \sin B + k \sin A\} \alpha$$
$$+ \{k \sin B - l(m - l \cos C) \sin B\} \beta$$
$$+ \{k \sin C + l(m - l \cos C) \sin C\} \gamma = 0,$$

and when $\alpha = 0$,

$$\frac{\beta}{\gamma} = \frac{(m - l \cos C)^2}{l^2 \sin B . \sin C};$$

therefore $k = \dfrac{l(m - l \cos C)\{(m - l \cos C)^2 - l^2 \sin^2 C\}}{(m - l \cos C)^2 + l^2 \sin^2 C}$,

and the equation becomes, for $TM$,

$$\sin B \{l^4 - 2lm(l^2 - m^2) \cos C - m^4\} \alpha$$
$$- 2l(m - l \cos C) \sin C \{l^2 \sin B . \sin C . \beta - (m - l \cos C)^2 \gamma\} = 0.$$

Again, a line passing through $T$ and perpendicular to $TM$ will be the directrix of the parabola.

Let $P = l^2 \sin C . \cos A + lm \sin A - m^2 \sin B$,
$Q = l \sin B (m - l \cos C)$,
$R = l \sin C (m - l \cos C)$.

Now the directrix may be represented by

$$(m + hn) \alpha + l\beta + hl\gamma = 0,$$

where

$$h = \frac{(m - l \cos C)\{P(l \cos C - m) + Q(l - m \cos C) + R(l \cos A + m \cos B)\}}{l \sin B \{Pl \sin C + Qm \sin C + R(m \sin B - l \sin A)\}},$$

therefore $m + hn$ = a fraction, the numerator of which is

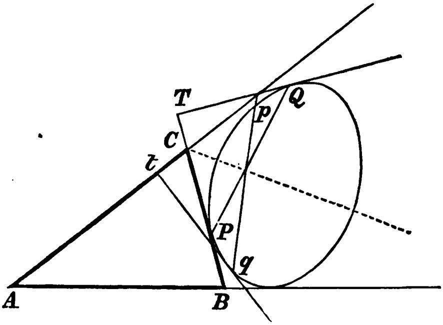
$$P\{l^2 \cos A + lm \cos B (1 + \cos C) - m^2 \cos B\}$$
$$+ Q\{l^2 \cos A + lm \cos B (1 - \cos C) + m^2 \cos A\}$$
$$+ R\{l^2 \cos A + lm (\cos B - \cos C) + m^2\},$$

and the denominator

$$\sin B \{Pl \sin C + Qm \sin C + R(m \sin B - l \sin A)\}.$$

211. Let a conic touch the three sides of a triangle $ABC$, viz. $BC$ externally and $AB$ and $AC$ produced; find the equa-

tions to the tangents $QT$, $qt$ respectively perpendicular to $\alpha = 0$ and $\beta = 0$, also find the equations to the chords of contact.



Now the equation to the conic thus enunciated will be

$$\sqrt{\{(-l\alpha)\}} + \sqrt{\{(m\beta)\}} + \sqrt{\{(n\gamma)\}} = 0 \text{............(1).}$$

Draw a tangent to the conic at right angles to $\alpha = 0$.

Let $$A\alpha + B\beta + C\gamma = 0 \text{......................(2)}$$

be the tangent required; then there will be two conditions

$$-\frac{l}{A} + \frac{m}{B} + \frac{n}{C} = 0 \text{......................(3),}$$

and $$A = B\cos C + C\cos B \text{..................(4).}$$

From (3) and (4) we obtain the equation

$$B^2 n\cos C + BC(n\cos B + m\cos C - l) + C^2 m\cos B = 0 \text{...(5),}$$

which must be a perfect square, and

$$l + 2\sqrt{(mn\cos B \,.\, \cos C)} - n\cos B - m\cos C = 0\,;$$

therefore, we have

$$\frac{B}{C} = -\sqrt{\left(\frac{m\cos B}{n\cos C}\right)}, \text{ or } \sqrt{(l)} = \sqrt{(n\cos B)} - \sqrt{(m\cos C)},$$

$$\frac{A}{C} = \sqrt{\left(\frac{l\cos B}{n}\right)},$$

and therefore the equation for $TQ$ is

$$\sqrt{(l\cos B \,.\, \cos C)}\,\alpha - \sqrt{(m\cos B)}\,\beta + \sqrt{(n\cos C)}\,\gamma = 0 \text{...(6).}$$

Again, to find the tangent at $t$, by the same kind of process, we get

$$\sqrt{(l\cos C)} - \sqrt{(-n\cos A)} + \sqrt{(m)} = 0,$$

and $$\sqrt{(l\cos C)} - \sqrt{(-n\cos A)} - \sqrt{(m)} = 0;$$

therefore, for the equation to the tangent at $t$ and at right angles to $\beta = 0$, we have

$$\sqrt{(l\cos A)}\,\alpha + \sqrt{(m\cos A\,.\cos C)}\,\beta - \sqrt{(-n\cos C)}\,\gamma = 0\ldots(7).$$

Next, let it be required to find the chords $PQ$ and $pq$.

If in the equation

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma + 2nl\gamma\alpha + 2lm\alpha\beta = 0\ldots(8)$$

we make $\alpha = 0$, then $\pm(m\beta - n\gamma) = 0$ will represent $AP$; therefore the equation to the chord $PQ$ will have the form

$$\lambda\alpha \pm (m\beta - n\gamma) = 0,$$

squaring and subtracting it from (8), we get

$$\alpha\{(l^2 - \lambda^2)\,\alpha + 2m(l-\lambda)\,\beta + 2n(l+\lambda)\,\gamma\} = 0;$$

therefore $\alpha = 0$ and $\alpha + \dfrac{2m}{l+\lambda}\beta + \dfrac{2n}{l-\lambda}\gamma = 0$ represent the tangents at the extremities of the chord $PQ$; therefore we have, for the chord $PQ$,

$$(m\cos C - n\cos B)\,\alpha \pm (m\beta - n\gamma) = 0\ldots\ldots\ldots(9).$$

By a similar process we obtain the equations to the tangents at the extremities of $pq$, viz.

$$\beta = 0 \text{ and } \frac{2l}{m+\lambda}\alpha + \beta - \frac{2n}{m-\lambda}\gamma = 0;$$

therefore we get for the chord $pq$ the equation

$$l\alpha + \sqrt{(m)}\,\{\sqrt{(l\cos C)} + \sqrt{(-n\cos A)}\}\,\beta + n\gamma = 0\ldots(10).$$

212. Let it be required to find the normals at the points $P$ and $Q$ in the foregoing.

Now the normal at $P$ will be perpendicular to $\alpha = 0$, and as it passes through the intersection of the lines represented by

$$(n\cos B - m\cos C)\,\alpha + m\beta - n\gamma = 0,$$

and $$\alpha = 0.$$

The equation to the normal at $P$ will be

$$(m \cos C - n \cos B)\,\alpha + m\beta - n\gamma = 0.$$

Also the normal at $Q$ will be parallel to $\alpha = 0$, and will pass through the intersection of the lines represented by

$$\sqrt{(l \cos B \,.\, \cos C)}\,\alpha - \sqrt{(m \cos B)}\,\beta + \sqrt{(n \cos C)}\,\gamma = 0,$$

and $$(n \cos B - m \cos C)\,\alpha + m\beta - n\gamma = 0\,;$$

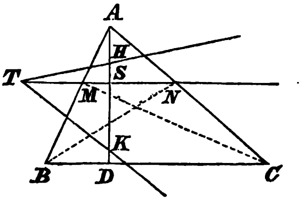
therefore the equation to the normal will be

$$\{2\,(m \sin C + n \sin B)\,\sqrt{(\cos B \,.\, \cos C)} + \sqrt{(mn)} \sin A\}\,\alpha - \sqrt{(mn)}\,(\beta \sin B + \gamma \sin C) = 0\,;$$

therefore the equation to the line passing through the intersections of those normals and tangents will be

$$\{3\,\sqrt{(\cos B \,.\, \cos C)}\,(m \sin C + n \sin B) + 2 \sin A\,\sqrt{(mn)}\}\,\alpha - \frac{n \sin B + m \sin C}{\sqrt{(n \cos B)} - \sqrt{(m \cos C)}}\,\{\sqrt{(m \cos B)}\,\beta - \sqrt{(n \cos C)}\,\gamma\} = 0.$$

213. If a conic section be inscribed in a triangle and have one of its foci on the perpendicular from the vertex upon the base. To determine the focus, the axes, and directrix.



Let the usual notation be adopted. But by Question 74, Chap. II., $h^{\frac{1}{2}} = \pm \sqrt{\left(\dfrac{\cos B}{\cos C}\right)}$, and since the equation to $AD$ is

$$\beta \cos B - \gamma \cos C = 0\,;$$

therefore the equation to the tangent where this line cuts the curve is

$$\pm\, l \sqrt{(\cos B \,.\, \cos C)}\,\alpha - \{\pm \sqrt{(m \cos C)} + \sqrt{(n \cos B)}\}\,\sqrt{(m \cos B)}\,\beta - \{\sqrt{(m \cos C)} \pm \sqrt{(n \cos B)}\}\,\sqrt{(n \cos C)}\,\gamma = 0,$$

so that the equation to the tangent at the point of contact $H$ will be

$$l\sqrt{(\cos B\,.\cos C)}\,\alpha - \{\sqrt{(m\cos C)}+\sqrt{(n\cos B)}\}$$
$$\times\{\sqrt{(m\cos B)}\,\beta+\sqrt{(n\cos C)}\,\gamma\}=0\,.........(1),$$

and that at the point of contact $K$

$$l\sqrt{(\cos B\,.\cos C)}\,\alpha - \{\sqrt{(m\cos C)}-\sqrt{(n\cos B)}\}$$
$$\times\{\sqrt{(m\cos B)}\,\beta-\sqrt{(n\cos C)}\,\gamma\}=0\,.........(2).$$

By subtraction, we get

$$\beta\cos B+\gamma\cos C=0\,..................(3),$$

which is a line upon which these two tangents meet.

Then the line $TS$ perpendicular to $AD$ will pass through the focus at $S$ and can be represented by

$$l\sqrt{(\cos B\,.\cos C)}\,\alpha$$
$$-\,[\{\sqrt{(m\cos C)}+\sqrt{(n\cos B)}\}\sqrt{(m\cos B)}-\lambda\cos B]\,\beta$$
$$-\,[\{\sqrt{(m\cos C)}+\sqrt{(n\cos B)}\}\sqrt{(n\cos C)}-\lambda\cos C]\,\gamma=0,$$

and subjecting this equation to the condition that it should be parallel to $BC$, we get

$$\lambda=\frac{\{\sqrt{(m\cos C)}+\sqrt{(n\cos B)}\}\{\sin B\sqrt{(n\cos C)}-\sin C\sqrt{(m\cos B)}\}}{\sin(B-C)}.$$

Hence the equation to $TS$ will become

$$\sin(B-C)\,l\alpha-(m\cos C-n\cos B)(\beta\sin B+\gamma\sin C)=0...(4);$$

therefore, for $CM$ $\quad l\sin(B-C)\,\alpha-(m\cos C-n\cos B)\,\beta\sin B=0,$

$BN$ $\quad l\sin(B-C)\,\alpha-(m\cos C-n\cos B)\,\gamma\sin C=0.$

And by subtraction we see that $CM$ and $BN$ intersect in the line which joins the vertex $A$ with the middle point of the opposite side

$$\beta\sin B-\gamma\sin C=0,$$

which is a well known property.

Now the axis-major can be obtained from the equation

$$l\sin(B-C)\,\alpha-\{(m\cos C-n\cos B)\sin B-\mu\cos B\}\,\beta$$
$$-\{(m\cos C-n\cos B)\sin C+\mu\cos C\}\,\gamma=0,$$

when this line passes through the center of the conic, it must be satisfied by the coordinates of the center (*vide* Question 89, p. 33). Hence

$$\mu = -\frac{\sin A\{l(m\sin C - n\sin B) + (m\cos C - n\cos B)(n\sin B + m\sin C)\}}{l\sin(B-C) + \sin A\,(m\cos C - n\cos B)};$$

therefore the equation to the axis-major will be

$$\begin{aligned} & l\sin(B-C)\,\{l\sin(B-C) + \sin A\,(m\cos C - n\cos B)\}\,\alpha \\ & - [l\{m(\cos B - \cos C)^2 - 2n\sin^2 B.\cos B.\cos C\} \\ & \qquad\qquad + m\sin^2 A\,(m\cos C - n\cos B)]\,\beta \\ & - [l\{n(\cos B - \cos C)^2 - 2m\sin^2 C.\cos B.\cos C\} \\ & \qquad\qquad - n\sin^2 A\,(m\cos C - n\cos B)]\,\gamma = 0. \end{aligned}$$

214. Let the focus of the ellipse be on the line joining $A$ with the bisection of the opposite side.

Here $$h = \frac{\sin B}{\sin C};$$

therefore the tangent at $H$ will be

$$\begin{aligned} & l\sqrt{(\sin B.\sin C)} - \{\sqrt{(m\sin C)} + \sqrt{(n\sin B)}\}\sqrt{(m\sin B)}\beta \\ & \qquad - \{\sqrt{(m\sin C)} + \sqrt{(n\sin B)}\}\sqrt{(n\sin C)}\gamma = 0\ldots(1). \end{aligned}$$

The tangent at $K$ will be

$$\begin{aligned} & l\sqrt{(\sin B.\sin C)} - \{\sqrt{(m\sin C)} - \sqrt{(n\sin B)}\}\sqrt{(m\sin B)}\beta \\ & \qquad + \{\sqrt{(m\sin C)} - \sqrt{(n\sin B)}\}\sqrt{(n\sin C)}\gamma = 0\ldots(2). \end{aligned}$$

By subtraction

$$\beta\sin B + \gamma\sin C = 0,$$

which shews that the pole of $HK$ is on a line drawn through $A$ and parallel to the base $BC$.

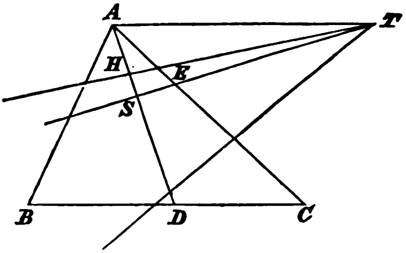
Since the equation to the conic is

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma\alpha - 2lm\alpha\beta = 0,$$

if $E$ and $F$ be the points of contact, the chord will be

$$l\alpha - m\beta - n\gamma = 0,$$

and if $S$ be the position of the focus, the equation to $ST$, since $AD$ is a focal chord, will be



$$\begin{aligned}&\alpha \sin^2 A \,.\, \sin(B-C)\\ &+\beta\{\sin A \,.\, \sin B \,.\, \sin(B-C)\\ &\qquad + l(m\sin C + n\sin B)(\sin C + \sin B \,.\, \cos A)\}\\ &+\gamma\{\sin A \,.\, \sin C \,.\, \sin(B-C)\\ &\qquad + l(m\sin C + n\sin B)(\sin B + \sin C \,.\, \cos A)\} = 0.\end{aligned}$$

Now call the coefficients of $\alpha$, $\beta$, $\gamma$; $U$, $V$, $W$, then the equation to some line passing through the focus can be represented by

$$U\alpha + (V + \lambda\sin B)\,\beta - \lambda\gamma\sin C = 0.$$

But if that line pass through the center also, it must be satisfied by the equations

$$\frac{\alpha}{R} = \frac{\beta}{Q} = \frac{\gamma}{P}.$$

From which we get

$$\lambda = \frac{RU + VP}{Q\sin C - P\sin B};$$

Therefore we have for the equation to the axis-major

$$\begin{aligned}&U(Q\sin C - P\sin B)\,\alpha + (VQ\sin C + RU\sin B)\,\beta\\ &\qquad\qquad - (PU + VP)\sin C \,.\, \gamma = 0.\end{aligned}$$

215. Find the equation to the radical axis of the inscribed circle and the circle through the middle points of the sides of a triangle

$$\frac{\alpha\cos\frac{1}{2}A}{\sin\frac{1}{2}(B-C)} + \frac{\beta\cos\frac{1}{2}B}{\sin\frac{1}{2}(C-A)} + \frac{\gamma\cos\frac{1}{2}C}{\sin\frac{1}{2}(A-B)} = 0.$$

216. Find the locus of the pole of a given line

$$l\alpha + m\beta + n\gamma = 0,$$

with regard to a conic which passes through four fixed points $\alpha'$, $\pm\beta'$, $\pm\gamma'$,

$$\frac{l\alpha'^2}{\alpha} + \frac{m\beta'^2}{\beta} + \frac{n\gamma'^2}{\gamma} = 0.$$

217. Find the locus of the pole of a given line

$$l\alpha + m\beta + n\gamma = 0,$$

with regard to a conic which touches four fixed lines

$$l'\alpha \pm m'\beta \pm n'\gamma,$$

$$\frac{l'^2\alpha}{l} + \frac{m'^2\beta}{m} + \frac{n'^2\gamma}{n} = 0.$$

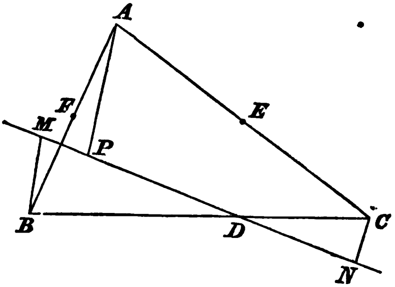
# CHAPTER III.

## TANGENTIAL COORDINATES.

218. In discussing the Method of Trilinear Coordinates, it has been understood that the position of a point is determined by its distances from three fixed lines; but in the system of tangential coordinates the position of any right line is determined by its distances from three given points.

Let $$AP = \alpha,$$
$$BM = \beta,$$
$$CN = \gamma.$$

Let $$BD = a_1, \quad DC = a_2;$$



therefore $$\frac{\beta}{a_1} = \frac{\gamma}{a_2},$$

which is the relation between the coordinates of any line passing through *D*.

It must be remembered that distances measured from *B* to *C* are positive; but when measured from *C* to *B* they are to be considered negative; in this case we assume *D* to be nearer *C* than *B*.

Now if *D* be the middle point of *BC*,

$$a_1 = -a_2;$$

therefore $$\beta + \gamma = 0.$$

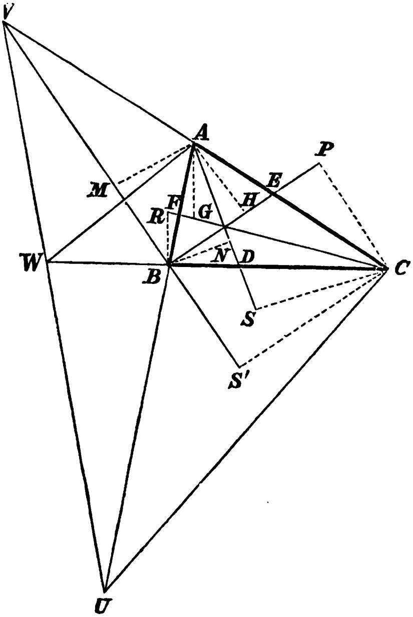
Similarly, if *E* be the middle point of *AC*,

$$\gamma + \alpha = 0,$$

also if *F* be the middle point of *AB*, we shall have

$$\alpha + \beta = 0.$$

219. Required to find the equation to the points where the angular bisectors cut the opposite sides.



Draw $AM$ perpendicular to $CF$,
$BN$ .................... $AD$,
$CP$ .................... $BE$ produced,
$BR$ .................... $CF$ produced,
$CS$ .................... $AD$ produced,
$AQ$ .................... $BE$,

since $AD$ bisects the angle $BAC$,

$$BN = c \sin \tfrac{1}{2}A, \quad \text{or } \beta = \phantom{-} c \sin \tfrac{1}{2}A,$$

$$CS = b \sin \tfrac{1}{2}A, \quad \text{or } \gamma = - b \sin \tfrac{1}{2}A\,;$$

for the point $D$, we have

$$\left.\begin{array}{lr} & \dfrac{\beta}{\gamma} = -\dfrac{c}{b}, \quad \text{or } b\beta + c\gamma = 0 \\ \text{Similarly, for } E \text{, we have} & c\gamma + a\alpha = 0 \\ \qquad\quad F, \text{ ........} & a\alpha + b\beta = 0 \end{array}\right\} \text{..............(1).}$$

Next let $BV$ be an external bisector,

$AM'$ perpendicular to $BV$,
$CS'$ .................... $BV$ produced;

therefore
$$AM' = \alpha = c \cos \tfrac{1}{2}B,$$
$$CS' = \gamma = a \cos \tfrac{1}{2}B\,;$$

therefore we have

$$\left.\begin{array}{lr} & \dfrac{\alpha}{\gamma} = \dfrac{c}{a}, \quad \text{or } c\gamma - a\alpha = 0 \text{ for the point } V \\ \text{Similarly} & a\alpha - b\beta = 0 \text{ for the point } W \\ \text{and} & b\beta - c\gamma = 0 \text{ for the point } U \end{array}\right\} \text{......(2).}$$

By adding together the equations (1), we have

$$\alpha + \beta + \gamma = 0,$$

shewing that $AD$, $BE$, $CF$ meet in a point.

By addition in equations (2) they vanish; hence $U$, $V$, $W$ range in a straight line.

220. The bisectors of the sides of a triangle meet in a point

$$\alpha + \beta + \gamma = 0.$$

This point is evidently satisfied by $\alpha + \beta = 0$ and $\gamma = 0$, the coordinates of the line joining $C$ with the middle point of $AB$.

Similarly, this point will be satisfied by the coordinates of the line joining $B$ with the middle point of $AC$, viz.

$$\gamma + \alpha = 0 \text{ and } \beta = 0;$$

the same may be said of the *line* joining the point $A$ with the middle point of $BC$, whose coordinates are

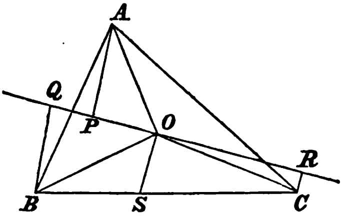
$$\beta + \gamma = 0 \text{ and } \alpha = 0.$$

We may observe here, that in tangential coordinates the condition that three straight lines meet in a point, is the same as the condition that three points lie in a straight line in trilinear coordinates and *vice versâ.*

221. If $O$ be any point within the triangle $ABC$, then the coordinates $\alpha$, $\beta$, $\gamma$ on any straight line passing through it, will be connected by the following equation (having regard to their proper signs)

$$\alpha \text{ area } BOC + \beta \text{ area } AOC + \gamma \text{ area } AOB = 0.$$

Let $ABC$ be the triangle, $O$ the given point within the triangle, $QR$ any line drawn through $O$.



Join $AO$, $BO$, and $CO$.

On $QR$ let fall the perpendiculars $AP$, $BQ$, $CR$. Draw $OS$ parallel to $BQ$ or $CR$.

Let $AP = -\alpha,\quad BQ = \beta,\quad CR = \gamma.$

Then $2AP\,.\,\text{area}\,BOC = AO\,.\,BO\,.\,CO\,.\,\sin AOQ\,.\,\sin BOC,$

and $2CR\,.\,\text{area}\,BOA$

$= AO.BO.CO.\cos OCR(\sin AOQ.\cos BOQ + \sin BOQ.\cos AOQ),$

$2BQ\,.\,\text{area}\,AOC$

$= AO.BO.CO.\cos OBQ(\sin AOR.\cos COR + \sin COR.\cos AOR),$

since $\cos OCR = \cos COS,\quad \cos BOQ = \sin BOS,$

$\sin BOQ = \cos OBQ,\quad \cos AOQ = -\cos AOR;$

therefore $2\{CR\,.\,\text{area}\,BOA + BQ\,.\,\text{area}\,AOC\}$

$= AO.BO.CO\,.\,\sin AOQ\,.\,\sin BOC.$

Hence, since $\alpha$ has been taken negative,

$$\alpha\,.\,\text{area}\,BOC + \beta\,.\,\text{area}\,AOC + \gamma\,.\,\text{area}\,AOB = 0,$$

as was required.

222. The bisectors of the sides of a triangle meet in a point, viz.

$$\alpha + \beta + \gamma = 0,$$

which is also the center of gravity; and since the point 0 is the center of gravity, by the last proposition the three triangles must have equal areas, hence the above form.

223. The center of the inscribed circle will be given by the equation

$$\alpha \sin A + \beta \sin B + \gamma \sin C = 0.$$

224. The center of the circumscribing circle will be given by the equation

$$\alpha \sin 2A + \beta \sin 2B + \gamma \sin 2C = 0.$$

225. The three perpendiculars let fall from the angular points of a triangle upon the opposite sides meet in the point represented by

$$\alpha \tan A + \beta \tan B + \gamma \tan C = 0.$$

226. The centers of the escribed circles will be determined by the equations

$$-\alpha\sin A+\beta\sin B+\gamma\sin C=0,$$
$$\alpha\sin A-\beta\sin B+\gamma\sin C=0,$$
$$\alpha\sin A+\beta\sin B-\gamma\sin C=0.$$

227. The middle points of the diagonals of a complete quadrilateral lie in a straight line.

Let $ABC$ be the triangle of reference, $D$ any point without it, forming the parallelogram $ABCD$, let $BA$ and $CD$ be produced to $E$ and $DA$, and $CB$ to $F$.

For $A, B, C, D$, we have

$$\alpha=0,\ \beta=0,\ \gamma=0;\ l\alpha+m\beta+n\gamma=0.$$

For the middle points of the three diagonals

$$\alpha+\beta=0,\quad (l+m+n)\beta+(l\alpha+m\beta+n\gamma)=0,$$

and $\quad ln(\alpha+\gamma)+\{(l+m+n)\beta+(l\alpha+m\beta+n\gamma)\}=0.$

The results in questions (223), (224), (225), (226) may be put under the forms respectively

$$a\alpha+b\beta+c\gamma=0,$$
$$a\alpha\cos A+b\beta\cos B+c\gamma\cos C=0,$$
$$\frac{a\alpha}{\cos A}+\frac{b\beta}{\cos B}+\frac{c\gamma}{\cos C}=0$$
$$-a\alpha+b\beta+c\gamma=0,$$
$$a\alpha-b\beta+c\gamma=0,$$
$$a\alpha+b\beta-c\gamma=0.$$

228. If any straight line cut the sides of a triangle, viz. $AB$ and $AC$ of the triangle of reference $ABC$ in $D$ and $E$, and if from $A$, $B$, $C$ perpendiculars be let fall $AP$, $BQ$, $CR$ upon $DE$.

If $\quad AP=-\alpha;\quad BQ=\beta,\quad CR=\gamma,$

the following relation will exist between the coordinates

$$b^2(\beta-\alpha)^2+c^2(\gamma-\alpha)^2-(b^2+c^2-a^2)(\beta-\alpha)(\gamma-\alpha)=b^2c^2\sin^2A,$$

or $a^2(\alpha-\beta)(\alpha-\gamma)+b^2(\beta-\gamma)(\beta-\alpha)$

$$+c^2(\gamma-\alpha)(\gamma-\beta)=4\,(\text{area})^2\,ABC.$$

A line at infinity may be considered as equidistant from the points $A$, $B$, $C$; therefore

$$\alpha=\beta=\gamma,$$

(*vide* Ferrers' *Trilinear Coordinates*, p. 124).

229. Apply tangential coordinates to prove that the centroid of a triangle, the intersection of the perpendiculars and the center of the circumscribed circle are in the same straight line.

Making use of the notation of Questions (222), (224), and (225), we find that the determinant

$$\begin{vmatrix} \sin 2A, & \sin 2P, & \sin 2C \\ \tan A, & \tan B, & \tan C \\ 1, & 1, & 1 \end{vmatrix} = 0.$$

230. The equation of the conic inscribed in a triangle is

$$l\alpha\beta+m\beta\gamma+n\gamma\alpha=0.$$

231. The equation of the conic circumscribing a triangle is

$$\sqrt{(l\alpha)}+\sqrt{(m\beta)}+\sqrt{(n\gamma)}=0,$$

or, in its rational state,

$$l^2\alpha^2+m^2\beta^2+n^2\gamma^2-2mn\beta\gamma-2nl\gamma\alpha-2lm\alpha\beta=0.$$

232. The equation to the inscribed circle is

$$(s-a)\,\beta\gamma+(s-b)\,\gamma\alpha+(s-c)\,\alpha\beta=0,$$

or

$$\beta\gamma\cot\tfrac{1}{2}A+\gamma\alpha\cot\tfrac{1}{2}B+\alpha\beta\cot\tfrac{1}{2}C=0.$$

233. The general equation of a parabola touching the three sides of the triangle $ABC$ is

$$l\beta\gamma+m\gamma\alpha-(l+n)\,\alpha\beta=0.$$

234. The equation to the circumscribing circle will be

$$a^4\alpha^2+b^4\beta^2+c^4\gamma^2-2b^2c^2\beta\gamma-2c^2a^2\gamma\alpha-2a^2b^2\alpha\beta=0,$$

which may be put in the simple form

$$\pm a\alpha^{\frac{1}{3}} \pm b\beta^{\frac{1}{3}} \pm c\gamma^{\frac{1}{3}} = 0.$$

235. If $l\alpha^2 + m\beta^2 + n\gamma^2 + 2l'\beta\gamma + 2m'\gamma\alpha + 2n'\alpha\beta = 0$ be the equation to a conic; the equation of the pole of the line $(\alpha'\beta'\gamma')$, with respect to the conic, will be

$$(l\alpha' + n'\beta' + m'\gamma')\,\alpha + (n'\alpha' + m\beta' + l'\gamma')\,\beta + (m'\alpha' + l'\beta' + n\gamma')\,\gamma = 0.$$

And, since the center is the pole at infinity when $\alpha = \beta = \gamma$, the equation of the center will be

$$(l + m' + n')\,\alpha + (l' + m + n')\,\beta + (l' + m' + n)\,\gamma = 0.$$

236. In Question (235) shew that if the conic be a parabola

$$l + m + n + 2l' + 2m' + 2n' = 0.$$

237. Every circle may be represented by the equation

$$a^2(\alpha - \beta)(\alpha - \gamma) + b^2(\beta - \gamma)(\beta - \alpha) + c^2(\gamma - \alpha)(\gamma - \beta) - (l\alpha + m\beta + n\gamma)^2 = 0.$$

# CHAPTER IV.

## QUADRILINEAR COORDINATES.

238. If $\alpha', \beta', \gamma', \delta'$ be the perpendiculars from any point on the four sides of a quadrilateral, they are connected by the relations

$$l\alpha' + m\beta' + n\gamma' + r\delta' = 0,$$

$$a\alpha' + b\beta' + c\gamma' + d\delta' = s,$$

where $l, m, n, r, a, b, c, d$ are constant functions of the sides and angles of the quadrilateral.

239. Shew that quadrilinear equations may be reduced to trilinear, by eliminating one of the variables; for instance

$$\delta = -(\alpha + \beta + \gamma).$$

What inference is to be derived from this transformation?

240. Shew that the general equation of the second degree may be of the form

$$\lambda\alpha\beta + \lambda'\gamma\delta + \mu\alpha\gamma + \mu'\beta\delta + \nu\alpha\delta + \nu'\beta\gamma = 0.$$

241. The straight line at infinity in its more general form has for its equation

$$(A + \kappa)\,\alpha + (B + \kappa)\,\beta + (C + \kappa)\,\gamma + (D + \kappa)\,\delta = 0.$$

242. Let $ABCD$ be a quadrilateral of reference, the equations to whose sides $AB$, $BC$, $CD$, $DA$ are respectively

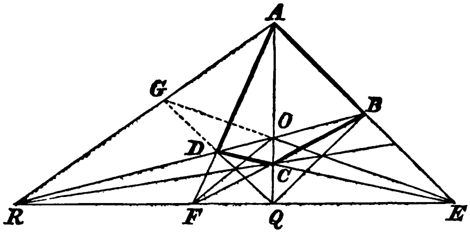
$$\alpha = 0,\ \ \beta = 0,\ \ \gamma = 0,\ \ \delta = 0;$$

constructing the figure as in the diagram, we have, for the three diagonals

$$BD\ldots\ldots\left\{\begin{array}{r}\alpha+\beta=0\\ \text{or } \gamma+\delta=0\end{array}\right\},$$

$$AC\ldots\ldots\left\{\begin{array}{r}\beta+\gamma=0\\ \delta+\alpha=0\end{array}\right\},$$

$$EF\ldots\ldots\left\{\begin{array}{r}\alpha+\gamma=0\\ \beta+\delta=0\end{array}\right\}.$$



Also for $OE\ldots\ldots\alpha-\gamma=0,\quad CR\ldots\ldots\beta-\gamma=0,$

$OF\ldots\ldots\beta-\delta=0,\quad BQ\ldots\ldots\alpha-\beta=0,$

$AR\ldots\ldots\alpha-\delta=0,\quad DQ\ldots\ldots\gamma-\delta=0,$

several interesting properties may be deduced from these equations, such as follow: viz.

*OF* and *BQ* produced will meet on the line *AR* produced,
*OE* and *CR* will intersect on *BQ*,
*OE* and *DQ* will meet on *AR*,
*OF* and *CR* will meet on *DQ*.

243. The general form for the equation to the conic circumscribing the quadrilateral *ABCD* will be

$$\alpha\gamma-\kappa\beta\delta=0.$$

244. For the conic passing through *ACEF*,

$$\alpha\beta-\kappa\gamma\delta=0.$$

245. For the conic passing through *BEFD*,

$$\alpha\delta-\kappa\beta\gamma=0.$$

246. Find the general equation of an inscribed conic free from the terms whose arguments are

$$\alpha^2,\ \beta^2,\ \gamma^2,\ \delta^2,$$

(*Messenger of Mathematics*, Vol. I., p. 197)

$$\lambda(\alpha\beta+\gamma\delta)+\mu(\alpha\gamma+\beta\delta)+\nu(\alpha\delta+\beta\gamma)=0,$$

$\lambda$, $\mu$, $\nu$ being subject to the relation

$$\sqrt{(\lambda)}+\sqrt{(\mu)}+\sqrt{(\nu)}=0.$$

247. Prove that the condition, that the line $\alpha+\beta+\gamma=0$ may touch the conic represented by

$$l^2\alpha^2+m^2\beta^2+n^2\gamma^2-2mn\beta\gamma-2nl\gamma\alpha-2lm\alpha\beta=0,$$

will be $$l+m+n=0,$$

square the equation to the line and add it to the equation to the conic, which being a perfect square, we get

$$\sqrt{\{(m^2+1)(n^2+1)\}}=1-mn,$$
$$\sqrt{\{(l^2+1)(n^2+1)\}}=1-nl,$$
$$\sqrt{\{(l^2+1)(m^2+1)\}}=1-lm,$$

from which we readily get the condition

$$l+m+n=0.$$

248. Find the polar of any point $(\alpha'\beta'\gamma'\delta')$ with respect to the conic whose equation is

$$f(\alpha,\beta,\gamma,\delta)=0.$$

249. If four points on a conic be given, shew that the polar of any fixed point passes through a fixed point.

250. If a conic circumscribing the quadrilateral $ABCD$ be represented by $\alpha\gamma-\kappa\beta\delta=0$, find the tangents at the angular points $A$, $B$, $C$, $D$.

251. (I.) If a quadrilateral be inscribed in a conic section, and another quadrilateral be described touching the conic in the angular points of the former one, the four interior diagonals of the two quadrilaterals meet in one point (0) and the two exterior diagonals coincide ($EF$).

(II.) If a quadrilateral be inscribed in a conic, the points of intersection of opposide sides and the points of intersection of tangents at opposite angles lie all in one straight line.

252. Prove that the circles circumscribing the triangles *FDC*, *EAD*, *FAB*, *EBC* pass all through one point.

253. Find the condition that the general equation of the second degree

$$\lambda\alpha\beta + \lambda'\gamma\delta + \mu\alpha\gamma + \mu'\beta\delta + \nu\alpha\delta + \nu'\beta\gamma = 0$$

should represent a parabola

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1, & 0 \\ 1, & 0, & \lambda, & \mu, & \nu, & A \\ 1, & \lambda, & 0, & \nu', & \mu', & B \\ 1, & \mu, & \nu', & 0, & \lambda', & C \\ 1, & \nu, & \mu', & \lambda', & 0, & D \\ 0, & A, & B, & C, & D, & 0 \end{vmatrix} = 0.$$

254. Through any four points on a parabola another parabola can always be drawn, unless the four points be symmetrically situated at the extremities of two parallel chords perpendicular to the axis.

255. If $\lambda(\alpha\beta + \gamma\delta) + \mu(\alpha\gamma + \beta\delta) + \nu(\alpha\delta + \beta\gamma) = 0$

be the equation to a conic inscribed in a quadrilateral; the only parabola which can be inscribed in it will have for its equation the form

$$(A - B)^2 (C - D)^2 (\alpha\beta + \gamma\delta) + (A - C)^2 (B - D)^2 (\alpha\gamma + \beta\delta)$$
$$+ (A - D)^2 (B - C)^2 (\alpha\delta + \beta\gamma) = 0.$$

256. If the two sides *BC*, *B'C'* of any hexagon *ABCA'B'C'* intersect on the diagonal produced, and the two sides *CA'*, *C'A* on the diagonal *BB'* produced, then will the remaining sides *AB*, *A'B'* intersect on the diagonal *CC'* produced (*vide Educational Times*, Vol. II., p. 58, the Reprint).

# CHAPTER V.

## TRIANGULAR AND TETRAHEDRAL COORDINATES.

257. In triangular coordinates, the position of a point $P$ is considered as determined by the ratios of the areas of the triangles $PBC$, $PCA$, and $PAB$ to the triangle of reference $ABC$, *i.e.* if

$$\frac{\text{area}\,(PBC)}{\text{area}\,(ABC)} = \alpha, \quad \frac{\text{area}\,(PCA)}{\text{area}\,(ABC)} = \beta, \quad \frac{\text{area}\,(PAB)}{\text{area}\,(ABC)} = \gamma,$$

then $$\alpha + \beta + \gamma = 1.$$

258. The three lines which join the angular points with the bisections of the opposite sides meet in a point.

The three straight lines may be represented respectively by

$$\beta - \gamma = 0, \quad \gamma - \alpha = 0, \quad \alpha - \beta = 0.$$

259. The three internal angular bisectors of a triangle meet in a point. Their respective equations are

$$\beta \sin C - \gamma \sin B = 0, \quad \gamma \sin A - \alpha \sin C = 0, \quad \alpha \sin B - \beta \sin A = 0.$$

260. The three external bisectors of the angles of a triangle meet the three sides produced in three points which range in a straight line.

261. The perpendiculars from the angular points on the opposite sides meet in a point and are represented by

$$\cot B - \gamma \cot C = 0, \quad \gamma \cot C - \alpha \cot A = 0, \quad \alpha \cot A - \beta \cot B = 0.$$

262. The distance between any two points will be represented by

$$\{bc \cos A\,(\alpha - \alpha')^2 + ac \cos B\,(\beta - \beta')^2 + ab \cos C\,(\gamma - \gamma')^2\}^{\frac{1}{2}}.$$

263. The condition that $l\alpha + m\beta + n\gamma = 0$ may be parallel to $l'\alpha + m'\beta + n'\gamma = 0$ is

$$mn' - m'n + nl' - n'l + lm' - l'm = 0.$$

264. The condition that the two lines, whose equations are

$$l\alpha + m\beta + n\gamma = 0 \text{ and } l'\alpha + m'\beta + n'\gamma,$$

should be perpendicular to each other is

$$ll'a^2 + mm'b^2 + nn'c^2$$
$$= (mn' + m'n)\,bc \cos A + (nl' + n'l)\,ac \cos B + (lm' + l'm)\,ab \cos C.$$

265. The perpendicular distance of the line $l\alpha + m\beta + n\gamma = 0$ from the point $(\alpha, \beta, \gamma)$ is

$$\frac{(l\alpha + m\beta + n\gamma)\; 2 \text{ area } (ABC)}{\{(l - m)(l - n)\,a^2 + (m - n)(m - l)\,b^2 + (n - l)(n - m)\,c^2\}^{\frac{1}{2}}}.$$

266. The volumes of the corners of the parallelopiped cut off by the faces of the tetrahedron are equal.

267. The center of gravity of the tetrahedron is the same as that of the parallelopiped.

268. If the lines joining the middle points of opposite edges of the tetrahedron are right angles, the opposite edges are equal.

269. The sum of the squares of the edges of the tetrahedron is equal to four times that of the lines joining the middle points of the opposite edges.

270. If two pairs of opposite edges be respectively at right angles, the third pair will also be at right angles to each other.

271. The triangular coordinates of a point remain unaltered, when the point and the triangle of reference are projected orthogonally.

272. The equation to the ellipse of least area circumscribing the triangle of reference will be

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 0.$$

273. The equation to the greatest ellipse, which can be inscribed in the triangle, will be

$$\sqrt{(\alpha)} + \sqrt{(\beta)} + \sqrt{(\gamma)} = 0.$$

274. The general equation of the second degree

$$A\alpha^2 + B\beta^2 + C\gamma^2 + 2A'\beta\gamma + 2B'\gamma\alpha + 2C'\alpha\beta = 0$$

will represent an ellipse, parabola, or hyperbola, according as

$$A'^2 + B'^2 + C'^2 - 2B'C' - 2C'A' - 2A'B' - BC - CA - AB + 2AA' + 2BB' + 2CC'$$

is negative, zero, or positive.

275. The condition that the foregoing general equation may be the representative of a circle

$$\frac{B + C - 2A'}{a^2} = \frac{C + A - 2B'}{b^2} = \frac{A + B - 2C'}{c^2}.$$

276. The condition that it will be a rectangular hyperbola is

$$(A + A' - B' - C')\,a^2 + (B + B' - C' - A')\,b^2 + (C + C' - A' - B')\,c^2 = 0.$$

277. If $\rho_a, \rho_b, \rho_c$ be the radii of an ellipse $\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0$ drawn parallel to the sides of the triangle of reference, then

$$\frac{l\rho_a^2}{a^2} = \frac{m\rho_b^2}{b^2} = \frac{n\rho_c^2}{c^2}.$$

278. Given the radii of a conic parallel to the sides of the triangle of reference to find its area (*vide Messenger of Mathematics*, Vol. I., p. 223).

279. Given the equation to any surface in tetrahedral to find its equation in tangential coordinates (*vide Messenger of Mathematics*, Vol. I., p. 225).

280. Find the equation of the ellipsoid of minimum volume, circumscribed about the tetrahedron of reference (*vide Messenger of Mathematics*, Vol. I., p. 228).

281. Find the equation to the ellipsoid of greatest volume inscribed in the tetrahedron of reference (*vide Messenger of Mathematics*, Vol. I., p. 228).

282. The equation to the circumscribing conicoid whose radii parallel to the edges of the tetrahedron are respectively $R_{ab}$, &c. is

$$\Sigma\left(\frac{\overline{AB}^2}{R_{ab}^2}\,\alpha\beta\right) = 0.$$

283. Given three points by equations of the form

$$lx + my + nz = 0;$$

prove that the area of the triangle contained by them is

$$\frac{\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}}{(l_1 + m_1 + n_1)(l_2 + m_2 + n_2)(l_3 + m_3 + n_3)},$$

that of the triangle of reference being unity.

This is Question 1497, *Educational Times*, by Mr. Clifford; *vide* solution by Mr. Whitworth, in the June Number of that periodical, 1864.

# CHAPTER VI.

## RECIPROCAL POLES AND POLARS.

Two systems of points related to each other in such a manner, that to every point of one system, there is a corresponding line of the other and to each point of the second system a corresponding line of the first. The point and line are named in reference to each other, the *pole* and *polar*, or reciprocal polars.

The tangent at any point of a conic section can be regarded as the polar of this point, and conversely this point can be regarded as the pole of the tangent.

The polar of a point in reference to a line of the second degree is parallel to the diameter.

The lines joining the points of contact of two tangents to the same conic section is the polar of the point of intersection of these tangents, or the polar in respect of a point without a conic section is the chord of contact of the two tangents drawn from that point.

The point of intersection of two chords of a conic section is the pole of the line joining the points of intersection of both pairs of tangents at the extremities of these chords.

284. If the angular point of an angle move upon a straight line whilst the lines containing the angle touch a curve of the second degree, whereby the angle is in general varied, the chords of contact turn about the same point, the pole of the line upon which the angular point moves.

285. Conversely, if the chord of a curve of the second degree turn about a fixed point, the tangents at the extremities of the chords intersect in the same straight line the polar of the fixed point.

286. Let a polygon of any number of sides be inscribed in a conic section, and let tangents be drawn through its angular points, so that a polygon of the same number of sides may circumscribe the same conic section; the two polygons are reciprocal polars of one another.

287. If a chord be drawn through the focus of a conic section, and tangents be drawn to the curve at the extremities of the focal chord, these tangents will intersect in the directrix.

288. Let a quadrilateral be inscribed within a conic section and from its angular points, as points of contact, a quadrilateral be described about the conic section, then

(1) If four pairs of opposite sides be produced to meet, the four points of intersection will range in one line.

(2) The four diagonals of these quadrilaterals will pass through the same point.

(3) The point and line will be a pole and polar in reference to the conic section.

289. If the pairs of opposite sides $DA$, $BC$ and $AB$, $DC$ of a quadrilateral $ABCD$, inscribed in a conic section be produced to meet in $M$ and $L$, and if the diagonals meet in $Q$; then $ML$ is the polar of the pole $Q$, $MQ$ the polar of the pole $L$ and $QL$ the polar of the pole $M$.

290. A circle passes through the center of an equilateral hyperbola and through any two points $A$, $B$ to prove that it will also pass through the intersection of two lines, one drawn through $A$ parallel to the polar of $B$, and the other through $B$ parallel to the polar of $A$.

291. When three points, situated in the plane of an equilateral hyperbola, are such that each of them is the pole of the line joining the other two, the circle which passes through these three points will pass also through the center of the hyperbola.

292. Prove that the diameter terminated at the point of contact of a tangent to an equilateral hyperbola, bisects two of the four angles, formed by the two diameters which pass through the points of intersection of this tangent with the polars of the two foci.

293. Prove that the diameters terminating at the two extremities of a chord of an equilateral hyperbola form respectively equal angles with those which pass through the intersection of this chord with the polars of the foci.

294. The diagonals of the circumscribed quadrilateral will pass through the points in which the opposite sides of the inscribed one intersect.

295. If from a point $P$ in an ellipse perpendiculars $PG$, $PH$ be let fall on two equal conjugate diameters; then the diagonal of the parallelogram upon $PG$, $PH$ is normal to the ellipse at the point $P$.

296. A triangle $ABC$ inscribed in a conic section is such that the line bisecting one of its interior angles ($A$) is a normal to the curve; then the side ($BC$) opposite the bisected angle will pass through the pole of the normal.

297. The polar of a point at an infinite distance is the diameter of the system of parallel chords which are directed to that infinitely distant point.

298. A focus and two points of a conic section being given, prove that the locus of the point of intersection of the tangents at these points will be two straight lines passing through the focus and at right angles to each other.

299. A conic is described touching three given straight lines *BC*, *CA*, *AB*, so that the pair of tangents drawn to it from a given point *O*, are at right angles to each other. Prove that it will always touch another fixed straight line; and that if this straight line cut *BC*, *CA*, *AB* in *D*, *E*, *F* respectively, each of the angles *AOD*, *BOE*, *COF* is a right angle, and prove that the polar of *O* with respect to this conic will always touch a conic, of which *O* is the focus.

300. If the lines which bisect the angles between pairs of tangents to an ellipse be parallel to a fixed straight line, prove that the locus of the points of intersection of the tangents will be a rectangular hyperbola.

301. Three hyperbolas are drawn, whose asymptotes are the sides of a triangle *ABC* taken two and two, prove that the direction of their three common chords pass through the angular points *A*, *B*, *C*, and meet in a point, which will be the center of gravity of the triangle, if the hyperbolas touch one another.

302. A parabola touches one side of a triangle in its middle point, and the other two sides produced. Prove that the perpendiculars drawn from the angles of the triangle upon any tangent to the parabola are in harmonical progression.

# CHAPTER VII.

## MISCELLANEOUS PROPERTIES EXTRACTED FROM THE MESSENGER OF MATHEMATICS AND OTHER RESOURCES.

303. EVERY straight line meets any cubic, either in three real points, or in one real and two imaginary points.

304. The general equation of the third degree in trilinear coordinates may be written

$$\begin{aligned} \alpha\beta\gamma = {} & \alpha^2(l_1\alpha + m_3\beta + n_1\gamma) \\ & + \beta^2(l_2\alpha + m_2\beta + n_2\gamma) \\ & + \gamma^2(l_3\alpha + m_3\beta + n_3\gamma) = 0. \end{aligned}$$

305. The only cubic having *three* double points not in the same straight line, consists of three straight lines forming a triangle.

306. The only cubic having *two* double points, consists of a straight line intersecting a conic section.

307. The cubic represented by the equation

$$\alpha(P\beta^2 + Q\beta\gamma + R\gamma^2) + \beta\gamma(N\beta + M\gamma) = 0$$

has at $\beta = 0$, $\gamma = 0$, a double point, a cusp, or conjugate point, according as

$$Q^2 > = < 4PR.$$

308. If a triangle be inscribed in a cubic so as to have its sides parallel to the asymptotes of the cubic; and if the tangents at $B$ and $C$ intersect in $P$; those at $C$, $A$ in $Q$; those at $AB$ in $R$. The three straight lines $PA$, $QB$, and $RC$ will all meet in a point.

309. The algebraical sum of the reciprocals of the perpendiculars from any point on the three sides of the inscribed equilateral triangle is constant.

310. The volume of the rectangular parallelopiped contained by the perpendiculars from any point on the three sides of the inscribed equilateral triangle bears a constant ratio to the volume of the rectangular parallelopiped contained by the perpendiculars on the tangents at the angular points of this triangle.

311. If a cubic have three rectilinear asymptotes, they cut the curve in three points lying all on one straight line.

312. If a cubic have three asymptotes, one of which cuts it in a finite point, another must do so also.

313. If two and only two asymptotes cut the curve in finite points, the other asymptote is parallel to the straight line joining those points of intersection.

314. The equation of cubics, which have a cusp, can always be reduced to the form

$$L^2M = R^3.$$

The line joining any two points of the curve will have, for its equation,

$$\mu\mu'(\mu + \mu')\,L - (\mu^2 + \mu\mu' + \mu'^2)\,R + M = 0;$$

hence the equation for the tangent at any point is

$$2\mu^3A - 3\mu^2R + M = 0.$$

315. The locus of the intersection of a chord drawn through any point on a cubic, and tangents where it meets the curve may be represented by

$$\gamma\,(2\gamma L + 3R)^2 + 2LM = 0,$$

(Salmon's *Higher Curves*).

316. If a curve of the third order have a double point $A$, and be cut by any straight line in $B$, $C$, $D$; and if when $ABC$

is taken as the triangle of reference, the tangents at $A$ are represented by the equation

$$P\beta^2 + Q\beta\gamma + R\gamma^2 = 0,$$

the tangents at $B$ and $C$ by the equations

$$P\alpha + N\gamma = 0 \text{ and } M\beta + R\alpha = 0,$$

shew that the equation to the straight line $AD$ is

$$N\beta + M\gamma = 0,$$

and find the equation to the curve (*Educational Times*).

317. Find the locus of the middle points of a system of chords parallel to $l\alpha + m\beta + n\gamma = 0$ in the conic

$$\phi(\alpha\beta\gamma) = A\alpha^2 + B\beta^2 + C\gamma^2 + 2A'\beta\gamma + 2B'\alpha\gamma + 2C'\alpha\beta = 0.$$

The equation of the locus required will be

$$(mc - bn)\frac{d\phi}{d\alpha'} + (na - lc)\frac{d\phi}{d\beta'} + (lb - ma)\frac{d\phi}{d\gamma'} = 0.$$

318. Determine the foci of a conic section expressed by trilinear coordinates (*vide Quarterly Journal of Mathematics*, Vol. IV., p. 235; and Vol. V., p. 177).

319. Determine the trilinear equation to the axes of a conic section (*vide Quarterly Journal of Mathematics*, Vol. V., pp. 273, and 335).

320. Determine the equation to the axes of a conic section referred to areal coordinates (*vide Messenger of Mathematics*, Vol. II., p. 16).

321. If

$$f(\alpha\beta\gamma) = l\alpha^2 + m\beta^2 + n\gamma^2 + 2l'\beta\gamma + 2m'\gamma\alpha + 2n'\alpha\beta = 0$$

be the equation to a conic in trilinear coordinates, the equation to the asymptotes will be

$$\begin{aligned}\{mn - l'^2 + nl - m'^2 + lm - n'^2 \\ + 2(m'n' - ll' + n'l' - mm' + l'm' - nn')\}\, f(\alpha, \beta, \gamma) \\ = (lmn + 2l'm'n' - ll'^2 - mm'^2 - nn'^2)(\alpha + \beta + \gamma)^2.\end{aligned}$$

322. Prove that the trilinear equation of the curve

$$(x^2 + y^2)^2 - 2ax(x^2 + 9y^2) + 27a^2y^2 = 0$$

is $$\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma) = 0,$$

the angular points of the triangle of reference being the origin, and the two points on the curve for which $x = \frac{9a}{4}$ (*Cambridge Problems*, 1865).

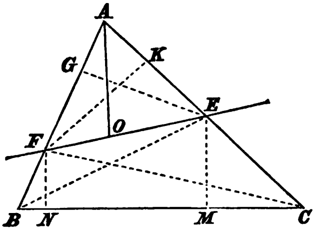
# CHAPTER VIII.

## APPENDIX.

323. QUESTION 50 has been discussed in the *Educational Times, vide* Question 1385, July, 1863.

It may at the same time be observed that it forms a good example for finding the equation to a line passing through two points.

Having constructed the figure, as in diagram, let the coordinates of $A$ and $O$ be respectively $\alpha'\beta'\gamma'$ and $\alpha''\beta''\gamma''$. Then the form for the equation of such a line will be, Question 8, p. 2,



$$\alpha(\beta'\gamma''-\gamma'\beta'')+\beta(\gamma'\alpha''-\gamma''\alpha')+\gamma(\alpha'\beta''-\alpha''\beta')=0.$$

Here $\beta'=0$ and $\gamma'=0$, and the equation to the line becomes

$$\beta\gamma''-\gamma\beta''=0,$$

$$\text{and}\quad -\frac{\gamma''}{\beta''}=\frac{-\frac{1}{2}GE}{\frac{1}{2}KF}=\frac{-GE}{KF}=\frac{-\text{area}\,ABC}{\text{area}\,ABC}\cdot\frac{GE}{KF}$$

$$=\frac{-\{FN.BC+KF.AC\}.GE}{\{GE.AB+EM.BC\}.KF}$$

$$=\frac{-\left\{\dfrac{FN}{KF}.BC+AC\right\}}{AB+\dfrac{EM}{GE}.BC}$$

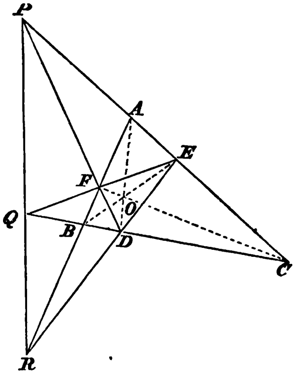
$$=\frac{ma-lb}{lc-an},\ \text{because}\ \frac{m}{l}=-\frac{FN}{KF}\ \text{and}\ \frac{n}{l}=-\frac{EM}{GE};$$

therefore the equation to $AO$, where $O$ is the middle point of $EF$, and $EF$ is represented by $l\alpha+m\beta+n\gamma=0$, will be

$$(ma-lb)\,\beta+(lc-na)\,\gamma=0.$$

324. In applying the properties enunciated in Question 11, p. 2, to Question 34.

Let $ABC$ be any triangle and $O$ any point whatever in it,



then if through this point, the lines $AD$, $BE$, $CF$ be drawn to the opposite sides, the triangle $DEF$ will be formed. Let $DF$ be produced to meet $AC$ in $P$, $EF$ to meet $BC$ in $Q$, and $DE$ to meet $AB$ in $R$. Then $P$, $Q$, $R$ will range in a straight line.

Now for $BC$ $\alpha = 0$, $AC$ $\beta = 0$, $AB$ $\gamma = 0$,

for
$$EF \quad \alpha + n\beta + \frac{1}{m}\gamma = 0,$$
$$DE \quad \frac{1}{n}\alpha + \beta + l\gamma = 0,$$
$$DF \quad m\alpha + \frac{1}{l}\beta + \gamma = 0.$$

Then for the intersection of $AB$ and $DE$, we have
$$\alpha + (n + \lambda)\beta + nl\gamma = 0 \text{..................} (1),$$
and for the intersection of $AC$ and $DF$
$$lm\alpha + \beta + (l + \lambda')\gamma = 0.$$

From (1)
$$\lambda = -\frac{\alpha + n\beta + ln\gamma}{\beta},$$

(2)
$$\lambda' = -\frac{lm\alpha + \beta + l\gamma}{\gamma},$$

substituting the coordinates of the intersection of $EF$ and $BC$ in $\lambda$ and $\lambda'$, *i.e.*
$$\alpha = 0, \quad \gamma = -mn\beta \text{ in } \lambda,$$
and
$$\alpha = 0, \quad \beta = -\frac{\gamma}{mn},$$
we have
$$\lambda = lmn^2 - n \text{ and } \lambda' = -\frac{lmn - 1}{mn},$$
and if $AD$, $BE$, and $CF$ pass through the same point $O$,
$$lmn = -1.$$
Hence
$$\lambda = -2n \text{ and } \lambda' = -2l,$$
substituting these values in (1) and (2), multiplying (2) by $n$ and recollecting that $lmn = -1$; in each case we obtain the equation
$$\alpha - n\beta + ln\gamma = 0,$$
which is the equation to the line $PQR$.

325. In Question 37, p. 10.

Find the equation to the straight line passing through the middle point of the side $BC$ of the triangle and parallel to the external bisector of the angle $A$.

Now the external bisector at $A$ is

$$\beta + \gamma = 0,$$

and any line passing through the middle point of the base will be

$$\lambda\alpha + \beta \sin B - \gamma \sin C = 0.$$

The condition that these two lines are parallel to each other is

$$\lambda = \frac{\sin A (\sin B + \sin C)}{\sin B - \sin C};$$

therefore

$$\alpha \sin A (\sin B + \sin C) + (\beta \sin B - \gamma \sin C)(\sin B - \sin C) = 0,$$

or $$(b + c)\, a\alpha + (b - c)(b\beta - c\gamma) = 0.$$

326. In Question 74, p. 30.

The equation to a tangent of the inscribed conic at the point where it is cut by the line $\beta - h\gamma = 0$, the conic being represented by $\sqrt{(l\alpha)} + \sqrt{(m\beta)} + \sqrt{(n\gamma)} = 0$ is

$$lh^{\frac{1}{2}}\alpha - \{\sqrt{(mh)} + \sqrt{(n)}\}\, m^{\frac{1}{2}}\beta - \{\sqrt{(mh)} + \sqrt{(n)}\}\, n^{\frac{1}{2}}h^{\frac{1}{2}}\gamma = 0 \ldots (1).$$

Here it must be recollected that the line cuts the curve in two points, and since the square root of $h$ will be $\pm h^{\frac{1}{2}}$; if we consider (1) the equation to the tangent at one point, the tangent at the other point will be

$$- lh^{\frac{1}{2}}\alpha - [-\sqrt{\{(mh)\}} + \sqrt{\{(n)\}}]\, m^{\frac{1}{2}}\beta$$
$$- [-\sqrt{\{(mh)\}} + \sqrt{(n)}]\, n^{\frac{1}{2}}(-h^{\frac{1}{2}})\gamma = 0 \ldots\ldots\ldots (2).$$

For example, in Cor. 1, p. 30, if a line be drawn from the vertex of the triangle of reference to the point of contact on the opposite side, the two tangents where this line cuts the conic will be determined from

$$h = \frac{n}{m},$$

*i.e.* $$h^{\frac{1}{2}} = \pm\left(\frac{n}{m}\right)^{\frac{1}{2}},$$

substituting these values respectively of $h^{\frac{1}{2}}$, we obtain the equation to the two tangents required, viz.

$$l\alpha - 2m\beta - 2n\gamma = 0 \text{ and } \alpha = 0.$$

Again, if the above conic had been cut by the angular bisector at *A*, the two tangents at the cutting points will be respectively represented by

$$l\alpha - \{m + \sqrt{(mn)}\}\beta - \{n + \sqrt{(mn)}\}\gamma = 0,$$

and $$-l\alpha + \{m - \sqrt{(mn)}\}\beta + \{n - \sqrt{(mn)}\}\gamma = 0.$$

By addition, we get

$$\beta + \gamma = 0,$$

which shews that these two tangents intersect each other in the external bisector at *A*.

327. Hence a solution of Question 16, p. 364, Walton's *Problems on Plane Coordinate Geometry.*

"From a point is drawn a pair of tangents to a conic section, and a line bisecting the interior angle between them is drawn cutting the corresponding polar in *O*: to prove that the intersection of a pair of tangents, belonging to any other polar passing through *O*, will lie in the line which bisects the exterior angle between the former pair of tangents."

Now we can easily conceive the figure, let *ABC* be the triangle of reference, *A* the given point, *AB* and *AC* the given tangents, *AD* the angular bisector. Let there be also an inscribed conic touching the sides *AB* and *AC* in *F* and *E* respectively; let *O* be the point where *EF* intersects *AD*.

(1) The tangents at the extremities of the line *EF* intersecting in the point *A* must also intersect on the line $\beta + \gamma = 0$.

(2) We have already proved that the tangents at the points where *AD* the angular bisector cuts the conic meet on the line $\beta + \gamma = 0$.

(3) Tangents at the extremities of all chords through *O* must meet on the line $\beta + \gamma = 0$, *vide* Garnier's *Analytical Geometry*, Art. 83, p. 161.

But we will prove this independently for any chord.

Now if the equation to the conic inscribed in the triangle be

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2ln\gamma\alpha - 2lm\alpha\beta = 0.$$

The equation to some line passing through the intersection of $\beta - \gamma = 0$ and $l\alpha - m\beta - n\gamma = 0$, or the point $O$ which we will call $(f, g, h)$ will be

$$l\alpha - (m - k)\,\beta - (n + k)\,\gamma = 0;$$

therefore $$\frac{f}{g} = \frac{m + n}{l} \text{ and } \frac{h}{g} = 1,$$

and the equation to the polar of $(f, g, h)$ will be

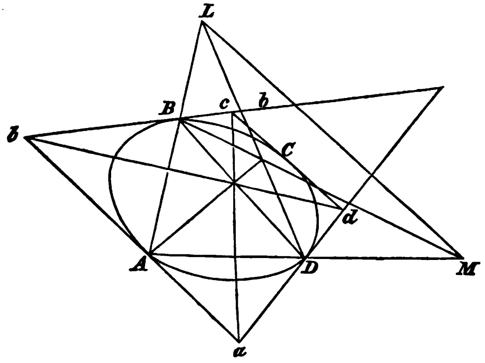
$$(l^2f - lmg - lnh)\alpha + (-lmf + m^2g - mnh)\beta + (-lnf - mng + n^2h)\gamma = 0.$$

By substitution

$$\beta + \gamma = 0,$$

which is the line required.

328. *Vide* Question 289, p. 105. If the pairs of opposite sides $DA$, $BC$ and $AB$, $DC$ of a quadrilateral $ABCD$ inscribed in a conic section be produced to meet in $M$ and $L$, and if the diagonals meet in $Q$; then $ML$ is the polar of the pole $Q$ and $MQ$ is the polar of the pole $L$ and $QL$ the polar of the pole $M$.



We can easily trace the figure by the following construction: Let $ABCD$ be the inscribed quadrilateral. Draw the diagonals $AC$ and $BD$ meeting in $Q$. Produce $CD$ and $BA$ to meet in $L$, $AD$ and $BC$ to meet in $M$. Through $Q$ draw $EFM$ and $LGH$, and if $ABC$ be taken as the triangle of reference, we shall have the following system of equations:

The equation to the curve being

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0.$$

For $BC$ $\alpha = 0$, for $AC$ $\beta = 0$, for $AB$ $\gamma = 0$,

for $AD$ $\beta + k\gamma = 0$, for $DC$ $(kn - m)\alpha - l\beta = 0$,

$$BD\ (kn - m)\alpha + lk\gamma = 0,$$

$$EF\ (kn - m)\alpha + l\beta + lk\gamma = 0,$$

$$GH\ (kn - m)\alpha - l\beta + lk\gamma = 0,$$

$$LM\ (kn - m)\alpha - l\beta - lk\gamma = 0.$$

Now if the equation to the curve had been

$$L\alpha^2 + M\beta^2 + N\gamma^2 + l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0.$$

The two tangents at the extremities of a chord passing through the point $(f, g, h)$ will intersect in the line (*vide* Ferrers' *Trilinear Coordinates*, Art. 8, p. 76)

$$\left(Lf + \frac{n}{2}g + \frac{m}{2}h\right)\alpha + \left(\frac{n}{2}f + Mg + \frac{l}{2}h\right)\beta + \left(\frac{m}{2}f + \frac{l}{2}g + Nh\right)\gamma = 0.$$

Now the line $EF$ passes through $(f, g, h)$ where $g = 0$, and

$$\frac{f}{h} = -\frac{lk}{hn - m},$$

and in the conic circumscribing the quadrilateral

$$L = 0, \quad M = 0, \quad N = 0;$$

therefore this equation becomes by substitution

$$(kn - m)\alpha - l\beta - kl\gamma = 0,$$

which is the equation to the line $LM$, and $ML$ is the polar of the pole $Q$.

Now let it be required to find the tangents at the extremities of $EF$ and $GH$ terminated by the curve.

First, for $EF$ we have shewn its equation to be

$$(kn - m)\,\alpha + l\beta + lk\gamma = 0.$$

Then the intersection of this line and $\beta + h\gamma$ will be in the line

$$(kn - m)\,\alpha + l\,(k - h)\,\gamma = 0,$$

and $\beta + h\gamma = 0$ meets the conic in

$$(hn - m)\,\alpha + lh\gamma = 0,$$

and since these two last equations must be identical, we have

$$\frac{kn - m}{k - h} = \frac{hn - m}{h},$$

hence $$h = \frac{m \pm \sqrt{(m^2 - kmn)}}{n};$$

therefore the tangent at $E$ will be (Question 71, p. 29)

$$n\,(m^2 - kmn)\,\alpha + lmn\beta + \{m + \sqrt{(m^2 - kmn)}\}^2\, l\gamma = 0 \ldots (1).$$

The tangent at $F$ will be

$$n\,(m^2 - kmn)\,\alpha + lmn\beta + \{m - \sqrt{(m^2 - kmn)}\}\; l\gamma = 0 \ldots (2).$$

These two tangents evidently meet on the line

$$\gamma = 0.$$

Hence $MQ$ is the polar of the pole $L$.

Again, the equation to $GH$ is

$$(kn - m)\,\alpha - l\beta + lk\gamma = 0,$$

and by a similar process, we get

$$h = \pm\sqrt{\left(\frac{km}{n}\right)};$$

therefore $$\{\pm\sqrt{(kmn)} - m\}^2\,\alpha + lm\beta + klm\gamma = 0,$$

which represent the two tangents respectively at $G$ and $H$, and they will intersect each other on

$$\alpha = 0,$$

and $QL$ is the polar of the pole $M$.

329. If a quadrilateral be inscribed in a conic section and another circumscribed about it so that the sides pass through the angular points of the former, then the diagonals of the

circumscribed quadrilateral will pass through the poin which the opposite sides of the inscribed one intersect.

We can readily conceive a quadrilateral $ABCD$ t inscribed in a conic section in such a manner that we call $ABC$ the triangle of reference.

At the points $A$, $B$, $C$, $D$ draw tangents $ab$, $bc$, $cd$, an Now for $AD$, we have

$$\beta + k\gamma = 0.$$

Hence by Question 71, p. 29, we have, for the tangent at

$$(nk - m)^2 \alpha + lm\beta + lnk^2\gamma = 0 \text{ ............ } (1$$

For the tangent at $A \quad n\beta + m\gamma = 0 \text{ ....................... } (2$

$$B \quad l\gamma + n\alpha = 0 \text{ ....................... } (3$$

$$C \quad m\alpha + l\beta = 0 \text{ ....................... } (4$$

therefore the diagonal $ac$ can be represented by

$$(nk - m)^2 \alpha + (lm + \lambda n)\beta + (lnk^2 + \lambda m)\gamma = 0 \text{ ... } (5$$

also the diagonal $ac$ can be represented by

$$(n + \lambda' m)\alpha + \lambda' l\beta + l\gamma = 0 \text{ ............... } (6$$

Since (5) and (6) are identical; therefore

$$\lambda' = \frac{lm + \lambda n}{lnk^2 + \lambda m} \text{ and } n + \lambda' m = \frac{n(lnk^2 + \lambda m) + m(lm + \lambda n}{lnk^2 + \lambda m}$$

whence we get $\lambda = -kl$,

and we have, for the equation to the diagonal $ac$,

$$(nk - m)\alpha - l\beta + kl\gamma = 0.$$

If $\gamma = 0$, then $(nk - m)\alpha - l\beta = 0$ which is the equation to hence the diagonal $ac$ passes through the intersection of and $CD$.

In the same way it may be shown that the diagonal $bd$ pass through the intersection of $AD$ and $BC$.

330. If $\alpha'\beta'\gamma'$ and $\alpha''\beta''\gamma''$ be two points on the conic re sented by $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$, the equation to the line joi them will be

$$\frac{l\alpha}{\alpha'\alpha''} + \frac{m\beta}{\beta'\beta''} + \frac{n\gamma}{\gamma'\gamma''} = 0 \text{ .................. } (1$$

The equation of the tangent to the conic represented by $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$ at the point $\alpha'\beta'\gamma'$ will be

$$\frac{l\alpha}{\alpha'^2} + \frac{m\beta}{\beta'^2} + \frac{n\gamma}{\gamma'^2} = 0 \text{ .....................} (2).$$

The forms are given in Salmon, Fourth Edition, p. 118.

Now by Question 8, p. 2, we have, for the equation to the line passing through the points $\alpha'\beta'\gamma'$ and $\alpha''\beta''\gamma''$,

$$\alpha(\beta'\gamma'' - \gamma'\beta'') + \beta(\gamma'\alpha'' - \gamma''\alpha') + \gamma(\alpha'\beta'' - \alpha''\beta') = 0;$$

therefore in order to find the values of

$$\frac{\beta'\gamma'' - \gamma'\beta''}{\gamma'\alpha'' - \gamma''\alpha'} \text{ and } \frac{\alpha'\beta'' - \alpha''\beta'}{\gamma'\alpha'' - \gamma''\alpha'}.$$

Since $\alpha'\beta'\gamma'$ and $\alpha''\beta''\gamma''$ are points on the conic, we must have

$$l\beta'\gamma' + m\gamma'\alpha' + n\alpha'\beta' = 0,$$

and

$$l\beta''\gamma'' + m\gamma''\alpha'' + n\alpha''\beta'' = 0,$$

which may be put under the form

$$\frac{l}{m}\beta'\gamma' + \frac{n}{m}\alpha'\beta' = -\gamma'\alpha',$$

and

$$\frac{l}{m}\beta''\gamma'' + \frac{n}{m}\alpha''\beta'' = -\gamma''\alpha''.$$

Hence we obtain

$$\frac{l\beta'\beta''}{m\alpha'\alpha''} = \frac{\beta'\gamma'' - \gamma'\beta''}{\gamma'\alpha'' - \gamma''\alpha'} \text{ and } \frac{n\beta'\beta''}{m\gamma'\gamma''} = \frac{\alpha'\beta'' - \alpha''\beta'}{\gamma'\alpha'' - \gamma''\alpha'};$$

therefore we have for the chord passing through $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$,

$$\frac{l\alpha}{\alpha'\alpha''} + \frac{m\beta}{\beta'\beta''} + \frac{n\gamma}{\gamma'\gamma''} = 0.$$

331. If this chord become a tangent, then $\alpha''\beta''\gamma''$ will respectively become $\alpha'\beta'\gamma'$, and the equation becomes

$$\frac{l\alpha}{\alpha'^2} + \frac{m\beta}{\beta'^2} + \frac{n\gamma}{\gamma'^2} = 0,$$

(*vide* Salmon, Fourth Edition, p. 122).

332. If $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$ be two points on the conic whose equation is $\sqrt{(l\alpha)}+\sqrt{(m\beta)}+\sqrt{(n\gamma)}=0$; then will the equation to the chord passing through those two points

$$\alpha\sqrt{(l)}\,\{\sqrt{(\beta'\gamma'')}+\sqrt{(\beta''\gamma')}\}+\beta\sqrt{(m)}\,\{\sqrt{(\gamma'\alpha'')}+\sqrt{(\gamma''\alpha')}\}$$
$$+\gamma\sqrt{(n)}\,\{\sqrt{(\alpha'\beta'')}+\sqrt{(\alpha''\beta')}\}=0.$$

This equation may be found in the same way as the last, by obtaining the values of $\sqrt{\left(\frac{l}{m}\right)}$ and $\sqrt{\left(\frac{n}{n}\right)}$ from the two equations

$$\sqrt{(l\alpha')}+\sqrt{(m\beta')}+\sqrt{(n\gamma')}=0,$$

and
$$\sqrt{(l\alpha'')}+\sqrt{(m\beta'')}+\sqrt{(n\gamma'')}=0,$$

where we have
$$\sqrt{\left(\frac{l}{m}\right)}=\frac{\sqrt{(\beta'\gamma'')}-\sqrt{(\gamma'\beta'')}}{\sqrt{(\gamma'\alpha'')}-\sqrt{(\gamma''\alpha')}},$$

and
$$\sqrt{\left(\frac{n}{m}\right)}=\frac{\sqrt{(\alpha'\beta'')}-\sqrt{(\alpha''\beta')}}{\sqrt{(\gamma'\alpha'')}-\sqrt{(\gamma''\alpha')}}.$$

But
$$\frac{\beta'\gamma''-\gamma'\beta''}{\gamma'\alpha''-\gamma''\alpha'}=\frac{\{\sqrt{(\beta'\gamma'')}+\sqrt{(\gamma'\beta'')}\}\,\{\sqrt{(\beta'\gamma'')}-\sqrt{(\gamma''\beta')}\}}{\{\sqrt{(\gamma'\alpha'')}+\sqrt{(\gamma''\alpha')}\}\,\{\sqrt{(\gamma'\alpha'')}-\sqrt{(\gamma''\alpha')}\}}$$
$$=\sqrt{\left(\frac{l}{m}\right)}\,\frac{\sqrt{(\beta'\gamma'')}+\sqrt{(\gamma'\beta'')}}{\sqrt{(\gamma'\alpha'')}+\sqrt{(\gamma''\alpha')}},$$

$$\frac{\alpha'\beta''-\alpha''\beta'}{\gamma'\alpha''-\gamma''\alpha'}=\frac{\{\sqrt{(\alpha'\beta'')}+\sqrt{(\alpha''\beta')}\}\,\{\sqrt{(\alpha'\beta'')}-\sqrt{(\alpha''\beta')}\}}{\{\sqrt{(\gamma'\alpha'')}+\sqrt{(\gamma''\alpha')}\}\,\{\sqrt{(\gamma'\alpha'')}-\sqrt{(\gamma''\alpha')}\}}$$
$$=\sqrt{\left(\frac{n}{m}\right)}\,\frac{\sqrt{(\alpha'\beta'')}+\sqrt{(\alpha''\beta')}}{\sqrt{(\gamma'\alpha'')}+\sqrt{(\gamma'\alpha')}}.$$

By substitution in the equation to the line passing through two points $\alpha'\beta'\gamma'$ and $\alpha''\beta''\gamma''$, we obtain the above equation required.

If we put in that equation $\alpha'\beta'\gamma'$ for $\alpha''\beta''\gamma''$ and divide off by $2\sqrt{(\alpha'\beta'\gamma')}$, we shall obtain the equation of the tangent at $\alpha'\beta'\gamma'$, viz.

$$\alpha\sqrt{(l\beta'\gamma')}+\beta\sqrt{(m\gamma'\alpha')}+\gamma\sqrt{(n\alpha'\beta')}=0.$$

333. Find an expression for the normal to the conic represented by $l\beta\gamma+m\gamma\alpha+n\alpha\beta=0$ at the point $\alpha'\beta'\gamma'$.

Let us represent the tangent by

$$A\alpha + B\beta + C\gamma = 0 \text{.....................} (1).$$

Now if a line be drawn through the point $A$ of the triangle of reference and perpendicular to the tangent which we will call $AT$, let its equation be

$$\beta + \lambda\gamma = 0 \text{.........................} (2).$$

The condition that (2) is perpendicular to (1) will be

$$\lambda = -\frac{C\cos A + A\cos C - B}{A\cos B + B\cos A - C};$$

therefore

$$(A\cos B + B\cos A - C)\,\beta - (C\cos A + A\cos C - B)\,\gamma = 0$$

is the equation to $AT$ which is perpendicular to the tangent.

Next, let it be required to draw a line through the point $\alpha'\beta'\gamma'$ and parallel to the line $AT$ which will be a normal to the curve, and, by Question 14, p. 3, note, we have

$$(A\cos B + B\cos A - C)\,\beta - (C\cos A + A\cos C - B)\,\gamma$$
$$+ k(\alpha\sin A + \beta\sin B + \gamma\sin C) = 0,$$

so if $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$, we have

$$-k = \frac{(A\cos B + B\cos A - C)\,\beta' - (C\cos A + A\cos C - B)\,\gamma'}{\alpha'\sin A + \beta'\sin B + \gamma'\sin C}.$$

Hence we shall have for the equation to the normal

$$k\alpha\sin A + (A\cos B + B\cos A - C + k\sin B)\,\beta$$
$$+ (k\sin C + B - C\cos A - A\cos C)\,\gamma = 0,$$

which becomes by substitution and dividing off by sin $A$

$$\{(C\cos A + A\cos C - B)\,\gamma' - (A\cos B + B\cos A - C)\,\beta'\}\,\alpha$$
$$+ \{(A\cos B + B\cos A - C)\,\alpha' - (C\cos B + B\cos C - A)\,\gamma'\}\,\beta$$
$$+ \{(C\cos B + B\cos C - A)\,\beta' - (A\cos C + C\cos A - B)\,\alpha'\}\,\gamma = 0$$
$$\text{......................} (3),$$

or it may be reduced to the following form by substituting the values of $A$, $B$, $C$ in (3) derived from the equation of the tangent where $A = \frac{l}{\alpha'^2}$, $B = \frac{m}{\beta'^2}$, $C = \frac{n}{\gamma'^2}$, viz.

$$\begin{aligned}
&\Big\{\frac{l}{\alpha'^2}(\gamma'\cos C-\beta'\cos B)-\frac{m}{\beta'^2}(\gamma'+\beta'\cos A)\\
&\qquad\qquad\qquad+\frac{n}{\gamma'^2}(\gamma'\cos A+\beta')\Big\}\alpha\\
+&\Big\{\frac{l}{\alpha'^2}(\alpha'\cos B+\gamma')\qquad+\frac{m}{\beta'^2}(\alpha'\cos A-\gamma'\cos C)\\
&\qquad\qquad\qquad-\frac{n}{\gamma'^2}(\alpha'+\gamma'\cos B)\Big\}\beta\\
+&\Big\{-\frac{l}{\alpha'^2}(\beta'+\alpha'\cos C)\qquad+\frac{m}{\beta'^2}(\beta'\cos C+\alpha')\\
&\qquad\qquad\qquad+\frac{n}{\gamma'^2}(\beta'\cos B-\alpha'\cos A)\Big\}\gamma=0.
\end{aligned}$$

From this equation the forms in Corollaries 1, 2, 3, Question 72, p. 29, may be readily obtained.

334. Find an expression for the normal at the point $\alpha'\beta'\gamma'$ on the conic represented by the equation

$$(l\alpha)^{\frac{1}{2}}+(m\beta)^{\frac{1}{2}}+(n\gamma)^{\frac{1}{2}}=0.$$

In the forms in Question (333) we must substitute for $A$, $B$, $C$ respectively $\sqrt{\left(\frac{l}{\alpha'}\right)}$, $\sqrt{\left(\frac{m}{\beta'}\right)}$, $\sqrt{\left(\frac{n}{\gamma'}\right)}$, and the equation to the normal becomes

$$\begin{aligned}
&\Big\{\sqrt{\left(\frac{l}{\alpha'}\right)}(\gamma'\cos C-\beta'\cos B)-\sqrt{\left(\frac{m}{\beta'}\right)}(\gamma'+\beta'\cos A)\\
&\qquad\qquad\qquad+\sqrt{\left(\frac{n}{\gamma'}\right)}(\gamma'\cos A+\beta')\Big\}\alpha\\
+&\Big\{\sqrt{\left(\frac{l}{\alpha'}\right)}(\alpha'\cos B+\gamma')\qquad+\sqrt{\left(\frac{m}{\beta'}\right)}(\alpha'\cos A-\gamma'\cos C)\\
&\qquad\qquad\qquad-\sqrt{\left(\frac{n}{\gamma'}\right)}(\alpha'+\gamma'\cos B)\Big\}\beta\\
+&\Big\{-\sqrt{\left(\frac{l}{\alpha'}\right)}(\beta'+\alpha'\cos C)\qquad+\sqrt{\left(\frac{m}{\beta'}\right)}(\beta'\cos C+\alpha')\\
&\qquad\qquad\qquad+\sqrt{\left(\frac{n}{\gamma'}\right)}(\beta'\cos B-\alpha'\cos A)\Big\}\gamma=0.
\end{aligned}$$

In order to verify this property, let us take a simple case for finding the normal to some point in the conic. Let the inscribed conic touch the sides *BC*, *AC*, and *AB* in the points *D*, *E*, *F*; therefore if we find the normal at the point *F*, whose coordinates we may assume to be $\alpha'\beta'\gamma'$; we can easily prove from the formula just obtained that it will take the well known form.

For if we make $\gamma' = 0$ and $\alpha' = \frac{m}{l}\beta'$, first multiplying the equation by $\sqrt{(\gamma')}$, we readily obtain the equation

$$l\alpha - m\beta + (l \cos B - m \cos A)\gamma = 0,$$

which is the equation to the normal at the point *F* in *AB*.

335. Let it be required to find an expression for the area of a triangle formed by three lines whose equations are referred to trilinear coordinates.

For an investigation of this interesting proposition the student is referred to the *Quarterly Journal of Pure and Applied Mathematics*, No. 26, p. 122, published by Longman and Co., June, 1865.

336. Let *ABC* be any triangle of reference, whose sides are represented, as usual, by $\alpha = 0$, $\beta = 0$, and $\gamma = 0$. If upon each of its sides equilateral triangles be described, it is required to find the equations to the lines passing through the angular points of the triangle of reference and through the centers of gravity of the equilateral triangles.

Having taken *ABC*, as usual, referred to trilinear coordinates, bisect *BC* in *D*, *AC* in *E*, and *AB* in *F*; and having described equilateral triangles respectively on the sides *BC*, *AC*, and *AB*, it is known that the perpendiculars from the angular points of the equilaterals upon the opposite sides will meet in their centers of gravity, since they bisect the opposite sides. Hence, if *G*, *H*, *K* be the respective centers of gravity; by the nature of the construction we shall have the angles *HAE*, *HCE*, *GCD*, *GBD*, *KBA*, *KAB* each equal to 30°; also, let *GN*, *GP* be perpendiculars on *AB* produced, and *AC*, *HQ*, and *HM* perpendiculars upon *AC* and *BA* produced, *KR* and *KS* perpendiculars upon *CA* *CB* produced respectively.

Now it will be readily seen that

$$DG = \frac{a}{2\sqrt{(3)}}, \quad GC = \frac{a}{\sqrt{(3)}},$$

$$EH = \frac{b}{2\sqrt{(3)}}, \quad AH = \frac{b}{\sqrt{(3)}},$$

$$KF = \frac{c}{2\sqrt{(3)}}, \quad BK = \frac{c}{\sqrt{(3)}},$$

First, let us find the equation to *AH*.

Now some line through *A* will be represented by

$$\beta + \lambda\gamma = 0;$$

therefore, when $\beta = -\frac{b}{2\sqrt{(3)}}$, $\gamma = \frac{b}{\sqrt{(3)}}\sin(A + 30)$;

therefore $\lambda = \frac{1}{2\sin(A + 30)}$;

therefore, equation to *AH* is

$$2\sin(A + 30)\,\beta + \gamma = 0.$$

Hence we have the following system of equations, viz. for

$$\begin{aligned}
&AH, && 2\sin(A + 30)\,\beta + \gamma = 0,\\
&CH, && \alpha + 2\sin(c + 30)\,\beta = 0,\\
&BG, && \gamma + 2\sin(B + 30)\,\alpha = 0,\\
&GC, && 2\sin(C + 30)\,\alpha + \beta = 0,\\
&KB, && 2\sin(B + 30)\,\gamma + \alpha = 0,\\
&AK, && \beta + 2\sin(A + 30)\,\gamma = 0.
\end{aligned}$$

Again, by p. 2, Ex. 8, if $\alpha'\beta'\gamma'$ and $\alpha''\beta''\gamma''$ be the coordinates of any two points, the equation to the line passing through them will be

$$\alpha(\beta'\gamma'' - \gamma'\beta'') + \beta(\gamma'\alpha'' - \gamma''\alpha') + \gamma(\alpha'\beta'' - \alpha''\beta') = 0.$$

Now we have for

$$\begin{aligned} &AH, \quad 2\sin(A+B)\,\beta' + \gamma' = 0,\\ &CH, \quad \alpha' + 2\sin(C+30)\,\beta' = 0,\\ &BG, \quad \gamma'' + 2\sin(B+30)\,\alpha'' = 0,\\ &CG, \quad 2\sin(C+30)\,\alpha'' + \beta'' = 0. \end{aligned}$$

Hence we have, for the equation to $HG$,

$$\{\surd(3)\sin B + 2\sin A\,.\sin C\}\,\alpha + \{\surd(3)\sin A + 2\sin B\,.\sin C\}\,\beta$$
$$-\sin C\,\{\sin C + \surd(3)\cos C\}\,\gamma = 0,$$

Similarly, we find for $HK$,

$$-\sin A\,\{\sin A + \surd(3)\cos A\}\,\alpha + \{\surd(3)\sin C + 2\sin A\,.\sin B\}\,\beta$$
$$+\{\surd(3)\sin B + 2\sin A\,.\sin C\}\,\gamma = 0,$$

and for $GK$,

$$\{\surd(3)\sin C + 2\sin A\,.\sin B\}\,\alpha - \sin B\,\{\sin B + \surd(3)\cos B\}\,\beta$$
$$+\{\surd(3)\sin A + 2\sin B\,.\sin C\}\,\gamma = 0.$$

It may also be shewn that the inclinations of these lines to one another are 60°, hence the triangle $GHK$ is equilateral.

337. In Question 97, p. 37, the three properties, viz. the equations—

(i) Circles described upon the sides of a triangle as diameters formed by joining the feet of the perpendiculars let fall from the angular points of a triangle upon the opposite sides.

(ii) Those described upon the sides of the triangle formed by joining the middle points of the sides of a triangle.

(iii) Those described upon the sides of the triangle formed by joining the points of intersection of the angular bisections.

In the reprint of the *Educational Times*, we have

(i) For the areal equation to the circle—

$$a^2yz + b^2zx + c^2xy$$
$$= \{(bc\cos A\sin^2 B)\,x + (ac\cos^3 B)\,y + (ab\cos C\sin^2 B)\,z\}\,(x+y+z).$$

For the trilinear equation—

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta$$
$$= (\alpha \cos A \sin^2 B + \beta \cos^3 B + \gamma \cos C \sin^2 B)(a\alpha + b\beta + c\gamma).$$

(ii) For the areal equation—

$$a^2yz + b^2zx + c^2xy$$
$$= \tfrac{1}{4}\{c(c + b\cos A)x + (ca\cos B)y + a(a + b\cos C)z\}(x + y + z).$$

For the trilinear—

$$4abc(a\beta\gamma + b\gamma\alpha + c\alpha\beta)$$
$$= \{ca(c\alpha + a\gamma) + abc(\alpha\cos A + \beta\cos B + \gamma\cos C)\}(a\alpha + b\beta + c\gamma).$$

(iii) For the areal equation—

$$a^2yz + b^2zx + c^2xy$$
$$= \frac{b^2c^2(a+b)(1+\cos A)x + c^2a^2\cos B(c+a)y + a^2b^2(a+b)(1+\cos C)z}{(a+b)(b+c)(c+a)}(x+y+z).$$

For the trilinear equation—

$$(a + b)(b + c)(c + a)(a\beta\gamma + b\gamma\alpha + c\alpha\beta)$$
$$= \{bc(a + b)(1 + \cos A)\alpha + ca\cos B(c + a)\beta$$
$$+ ab(a + b)(1 + \cos C)\gamma\}(a\alpha + b\beta + c\gamma).$$

FINIS.

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